Math 506 Homework 1

(1) Prove the Stokes' theorem on $D = [0,1] \times [0,1]$: For $P(x,y)$ and $Q(x,y) \in C^\infty(D)$,

$$\int_0^1 \int_0^1 \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \int_{\partial D} (Pdx + Qdy)$$

where ∂D is oriented counter-clockwise.

(2) Let $n \geq 3$ be an integer. Let $f(x)$ and $g(x)$ be polynomials with real coefficients such that the points $(f(1), g(1))$, $(f(2), g(2))$, ..., $(f(n), g(n))$ in $\mathbb{R}^2$ are the vertices of a regular n-gon in counter-clockwise order. Prove that at least one of $f(x)$ and $g(x)$ has degree greater than or equal to $n-1$.

(3) Show that if $f : \mathbb{C} \to \mathbb{C}$ is a continuous function such that f is analytic off the real axis, then f is an entire function.

(4) Compute the integrals

$$\int_0^\infty \sin(x^2)dx \text{ and } \int_0^\infty \cos(x^2)dx$$

(5) Suppose that $f : D = \{|z| < 1\} \to \mathbb{C}$ is holomorphic. Show that the diameter

$$d = \sup_{z,w \in D} |f(z) - f(w)|$$

of the image of f satisfies

$$2|f'(0)| \leq d$$

and the equality holds if and only if $f(z) = a + bz$ is linear.

(6) Weierstrass's theorem states that a continuous function on $[0,1]$ can be uniformly approximated by polynomials. Can every continuous function on the closed unit disc be approximated by polynomials in the variable z?

(7) Find $\mathrm{Aut}(D^*)$ where $D^* = \{0 < |z| < 1\}$ is the punctured disk.

(8) Morera's theorem states that if f is continuous in an open set $D \subset \mathbb{C}$ and $\int_T f(z)dz = 0$ for all triangles $T \subset D$, then f is analytic in D. Does the theorem still hold if we replace triangles by rectangles? What if we replace triangles by circles?