(1) Let $D \subset \mathbb{C}$ be an open set in the complex plane $\mathbb{C}$ and $f : D \to \mathbb{C}$ be a holomorphic function. Define $\varphi : D \times D \to \mathbb{C}$ by

$$
\varphi(z, w) = \frac{f(z) - f(w)}{z - w}
$$

if $z \neq w$ and $\varphi(z, z) = f'(z)$. Prove that $\varphi$ is continuous and for each fixed $w = w_0$, $\varphi(z, w_0)$ is holomorphic in $z$.

(2) Let $D = \{|z| < 1\}$ and $f : D \to \mathbb{C}$ be a holomorphic function on $D$. If $|f(z)| \leq 1$ for all $z \in D$, then $|f'(0)| \leq 1$.

(3) Let $D \subset \mathbb{C}$ be an open set and let $f_n : D \to \mathbb{C}$ be a sequence of analytic functions on $D$. Suppose that $\{f_n\}$ converges uniformly to a function $f : D \to \mathbb{C}$. Show that $f$ is analytic.

(4) Show that if $f : \mathbb{C} \to \mathbb{C}$ is a continuous function such that $f$ is analytic off $[0, 1]$, then $f$ is an entire function.

(5) What if we change $[0, 1]$ to $[0, \infty)$ in the previous problem?

(6) Let $n \geq 3$ be an integer. Let $f(x)$ and $g(x)$ be polynomials with real coefficients such that the points $(f(1), g(1)), (f(2), g(2)), \ldots, (f(n), g(n))$ in $\mathbb{R}^2$ are the vertices of a regular $n$-gon in counterclockwise order. Prove that at least one of $f(x)$ and $g(x)$ has degree greater than or equal to $n - 1$. 