Math 506 Midterm

(1) Compute the integral:
\[ \int_0^\infty \frac{\log x}{(1+x^2)^2} \, dx \]

(2) A map \( f : X \to Y \) is finite if \( f^{-1}(p) \) is finite for every point \( p \in Y \).
Show that if \( f : \mathbb{C} \to \mathbb{C} \) is finite and holomorphic, then \( f(z) \) must be a polynomial.

(3) A map \( f : X \to Y \) is proper if for every compact set \( D \subset Y \), \( f^{-1}(D) \) is compact.
Show that if \( f : \mathbb{C} \to \mathbb{C} \) is proper and holomorphic, then \( f(z) \) must be a polynomial.

(4) Show that \( D_1 = \{ 0 < |z| < 1 \} \) and \( D_2 = \{ 1 < |z| < 2 \} \) are not biholomorphic.

(5) A map \( f : D_1 \to D_2 \) is ramified at point \( p \in D_1 \) of index \( m > 0 \) if for every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for every \( y \in \{ 0 < |z-f(p)| < \delta \} \), there are exactly \( m + 1 \) distinct points \( x_1, x_2, \ldots, x_{m+1} \) in \( \{ 0 < |z-p| < \epsilon \} \) such that \( f(x_i) = y \) for \( i = 1, 2, \ldots, m+1 \). We call \( m \) the ramification index of \( f \) at \( p \). Show that if \( f \) is analytic at \( p \), then \( f \) is ramified at point \( p \) of index \( m \) if and only if \( f'(p) = f''(p) = \ldots = f^{(m)}(p) = 0 \) and \( f^{(m+1)}(p) \neq 0 \).

(6) Let \( f : D_1 \to D_2 \) be holomorphic, finite and onto where \( D_1 \) is a domain in \( \mathbb{C} \) (i.e. an open connected set). For every point \( q \in D_2 \), we let \( N(q) \) be the number of points in \( f^{-1}(q) \). Show that \( N(q) \) is a lower semicontinuous function on \( D_2 \), i.e., \( \{ q \in D_2 : N(q) > c \} \) is open for any \( c \in \mathbb{R} \).

(7) If \( f : D \to \mathbb{C} \) is analytic except for poles, show that the poles of \( f(z) \) cannot have a limit point in \( D \).

(8) Let \( f : D_1 \to D_2 \) be holomorphic and onto where both \( D_1 \) and \( D_2 \) are simply connected domains in \( \mathbb{C} \). Show that if \( f'(z) \) does not vanish, \( f \) is 1-1.