Math 506 Homework 1

(1) Let $D \subset \mathbb{C}$ be an open set in the complex plane $\mathbb{C}$ and $f : D \to \mathbb{C}$ be a holomorphic function. Define $\varphi : D \times D \to \mathbb{C}$ by

$$\varphi(z, w) = \frac{f(z) - f(w)}{z - w}$$

if $z \neq w$ and $\varphi(z, z) = f'(z)$. Prove that $\varphi$ is continuous and for each fixed $w = w_0$, $\varphi(z, w_0)$ is holomorphic in $z$.

(2) Let $D = \{|z| < 1\}$ and $f : D \to \mathbb{C}$ be a holomorphic function on $D$. If $|f(z)| \leq 1$ for all $z \in D$, then $|f'(0)| \leq 1$.

(3) Let $D \subset \mathbb{C}$ be an open set and let $f_n : D \to \mathbb{C}$ be a sequence of analytic functions on $D$. Suppose that $\{f_n\}$ converges uniformly to a function $f : D \to \mathbb{C}$. Show that $f$ is analytic.

(4) Show that if $f : \mathbb{C} \to \mathbb{C}$ is a continuous function such that $f$ is analytic off $[0, 1]$, then $f$ is an entire function.

(5) What if we change $[0, 1]$ to $[0, \infty)$ in the previous problem?

Math 506 Homework 2

(1) Compute

$$\int_0^\infty \frac{\log x}{(1 + x^2)^2} dx$$

(2) Let $M$ and $N$ be two topological spaces. Two continuous maps $f : M \to N$ and $g : M \to N$ are called homotopic, written as $f \sim g$, if there exists a continuous map $F : M \times [0, 1] \to N$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. And $M$ and $N$ are homotopic to each other if there exists two continuous maps $f : M \to N$ and $g : N \to M$ such that $f \circ g \sim 1_N$ and $g \circ f \sim 1_M$, where $1_M$ and $1_N$ are identity maps on $M$ and $N$, respectively. Show that $M = \mathbb{R}^2\backslash\{\text{two points}\}$ and $N = \{x^2 + y^2 = 1\} \cup \{(x - 2)^2 + y^2 = 1\} \subset \mathbb{R}^2$ are homotopic.

(3) Find the Laurent Series of $f(z) = (z - 1)^{-2}(z - 2)^{-1}$ in

(a) $|z| < 1$;
(b) $1 < |z| < 2$;
(c) $|z| > 2$.

(4) Let $D \subset \mathbb{C}$ be a connected open set. If $f : D \to \mathbb{C}$ is analytic except for poles, then the poles of $f$ cannot have a limit point.

(5) Let $\lambda > 1$ and show that the equation $\lambda - z - e^{-z} = 0$ has exactly one solution in the half plane $\{z : \text{Re } z > 0\}$.

(6) In the above problem, further show that the solution must be real. What happens to the solution as $\lambda \to 1$?

Math 506 Homework 3
We use the notation $O_n[[z_1, z_2, ..., z_n]]$ to denote the ring of germs of analytic functions in $n$ variables $z_1, z_2, ..., z_n$.

1. Let $f(x, y) \in O_2[[x, y]]$. Suppose that
   \[
   f(x, y) = (x - c_1 y)(x - c_2 y)...(x - c_m y) + \sum_{j+k>m} a_{jk} x^j y^k
   \]
   where $c_1, c_2, ..., c_m, a_{jk} \in \mathbb{C}$ are constants. We call $m$ the multiplicity of $f(x, y)$ at the origin. Show that if $c_1, c_2, ..., c_m$ are distinct numbers, then there exists $g_1(y), g_2(y), ..., g_m(y) \in O_1[[y]]$ such that
   \[
   f(x, y) = (x - g_1(y))(x - g_2(y))...(x - g_m(y))h(x, y)
   \]
   where $h(x, y) \in O_2[[x, y]]$ and $h(0, 0) \neq 0$.

2. Let $f(x, y) \in O_2[[x, y]]$. Suppose that $f(x, y)$ has multiplicity 2 at the origin, i.e., $f(x, y)$ is given by (0.1) with $m = 2$. Then there exists an automorphism $\phi : O_2[[x, y]] \to O_2[[x, y]]$ such that $\phi(f(x, y)) = x^2 - y^n$ for some $n \geq 2$.

3. Let $O_1([z])$ be the quotient field of $O_1([z])$. Then any finite extension of $O_1([z])$ is $O_1([z])$ itself and given by $\phi : O_1([z]) \to O_1([z])$ with $\phi(z) = z^n$ for some $n > 0$.

4. Let $\mathbb{C}[[z, w]]$ be the ring of formal power series. Prove Weierstrass Preparation Theorem on $\mathbb{C}[[z, w]]$. That is, let
   \[
   f(z, w) = w^m + \sum_{(i,j) \neq (0,m)} a_{ij} z^i w^j \in \mathbb{C}[[z, w]]
   \]
   Then $f(z, w) = g(z, w)h(z, w)$ where $g(z, w) = w^d + a_1(z)w^{d-1} + ... + a_d(z)$ is a Weierstrass polynomial and $h(0, 0) \neq 0$.

Math 506 Homework 4

1. Show that every holomorphic map from $f : \mathbb{P}^1 \to \mathbb{P}^1$ is given by $f(Z_0, Z_1) = (F_0(Z_0, Z_1), F_1(Z_0, Z_1))$ for some homogeneous polynomials $F_0$ and $F_1$.

2. Let $\Lambda_i = \{m\mu_i + n\nu_i : m, n \in \mathbb{Z}\} \subset \mathbb{C}$ be two nondegenerated lattices in $\mathbb{C}$ for $i = 1, 2$. Show that the corresponding complex tori $\mathbb{C}/\Lambda_1$ and $\mathbb{C}/\Lambda_2$ are isomorphic (i.e. biholomorphic) if $\Lambda_1 = a\Lambda_2$ for some constant $a \neq 0$. You may proceed as follows.
   (a) Every holomorphic map $\varphi : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ can be lifted to a holomorphic map on their universal coverings, i.e., we have the commutative diagram:

   \[
   \begin{array}{c}
   \mathbb{C} \xrightarrow{\phi} \mathbb{C} \\
   \downarrow \hspace{1cm} \downarrow \\
   \mathbb{C}/\Lambda_1 \xrightarrow{\varphi} \mathbb{C}/\Lambda_2
   \end{array}
   \]
(b) Show that $\phi$ in the above diagram is an isomorphism if $\varphi$ is an isomorphism, i.e., $\phi(z) = az + b$ for some $a \neq 0$ and $b \in \mathbb{C}$.

(c) Show that $\phi(\Lambda_1) = \Lambda_2$.

(3) Let $f : \mathbb{C} \to \mathbb{C}/\Lambda$ be a nonconstant holomorphic map from $\mathbb{C}$ to a complex torus $\mathbb{C}/\Lambda$. Show that $f$ is surjective.

(4) Let $E^0$ be the analytic set in $\mathbb{C}^2$ given by $E^0 = \{y^2 = x^3 - 1\}$. Show that $E^0$ is a complex submanifold of $\mathbb{C}^2$. Let $i : \mathbb{C}^2 \hookrightarrow \mathbb{P}^2$ be the embedding $i(x, y) = (x, y, 1)$ and $E$ be the closure of $i(E^0)$ in $\mathbb{P}^2$. Write down the defining equation for $E$ and show that $E$ is a complex submanifold of $\mathbb{P}^2$.

(5) Use (3) and (4) to show that there do not exist two nonconstant entire functions $y(t)$ and $x(t)$ such that $y^2(t) = x^3(t) - 1$. 