DUALITY IN LINEAR PROGRAMMING

We start with a linear programming problem in the following form:

**Problem 0.1.** Maximize \( f(x_1, x_2, ..., x_n) = \sum_{j=1}^{n} c_j x_j \) subjecting to the linear constraints

\[
\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \text{for } i = 1, 2, ..., m \text{ and } x_j \geq 0 \quad \text{for } j = 1, 2, ..., n.
\]

The dual of the above problem is

**Problem 0.2.** Minimize \( g(y_1, y_2, ..., y_m) = \sum_{i=1}^{m} b_i y_i \) subjecting to the linear constraints

\[
\sum_{i=1}^{m} a_{ij} y_i \geq c_j \quad \text{for } j = 1, 2, ..., n \text{ and } y_i \geq 0 \quad \text{for } i = 1, 2, ..., m.
\]

**Theorem 0.3.** \( f_{\text{max}} = g_{\text{min}} \)

It is not hard to see that \( f(x_1, x_2, ..., x_n) \leq g(y_1, y_2, ..., y_m) \) for any \( x_j \) and \( y_i \) satisfying the constraints. And hence \( f_{\text{max}} \leq g_{\text{min}} \). So it suffices to show that \( f_{\text{max}} \geq g_{\text{min}} \).

To make things simple, we assume that \( b_i \geq 0 \).

Let \( u_1, u_2, ..., u_m \geq 0 \) be the nonbasic variables. We convert Problem 0.1 to the standard form

**Problem 0.4.** Maximize \( f(x_1, x_2, ..., x_n) = \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} \alpha_i u_i + A \) subjecting to the linear constraints

\[
\sum_{j=1}^{n} a_{ij} x_j + u_i = b_i \quad \text{for } i = 1, 2, ..., m \text{ and } x_j, u_i \geq 0.
\]

At the beginning, \( \alpha_i = A = 0 \). During the process of simplex method, we always have \( \alpha_i \leq 0 \). Eventually, when \( c_j \leq 0 \) for all \( j \), \( f_{\text{max}} = A \).

We define the dual problem of 0.4 to be

**Problem 0.5.** Minimize \( h(y_1, y_2, ..., y_m) = \sum_{i=1}^{m} b_i y_i + A \) subjecting to the linear constraints

\[
\sum_{i=1}^{m} a_{ij} y_i \geq c_j \quad \text{for } j = 1, 2, ..., n \text{ and } y_i \geq 0 \quad \text{for } i = 1, 2, ..., m.
\]
We will show that $h_{\min} \geq g_{\min}$ during the process of simplex method. So eventually when $c_j \leq 0$, $f_{\max} = A \geq h_{\min} \geq g_{\min}$ and we are done.

After one step of simplex method, we subtract $k$-th row multiplied by some $\lambda \geq 0$ from the objective function. Then the new objective function becomes $f(x_1, x_2, ..., x_n) = \sum_{j=1}^{n} \hat{c}_j x_j + \sum_{i=1}^{m} a_{ij}u_i + \bar{A}$, where $\hat{c}_j = c_j - \lambda a_{kj}$ and $\bar{A} = A + \lambda b_k$.

Correspondingly, the dual problem becomes

**Problem 0.6.** Minimize $\bar{h}(\bar{y}_1, \bar{y}_2, ..., \bar{y}_m) = \sum_{i=1}^{m} b_i \bar{y}_i + \bar{A}$ subjecting to the linear constraints

\[
\sum_{i=1}^{m} a_{ij} \bar{y}_i \geq \hat{c}_j
\]

for $j = 1, 2, ..., n$ and $\bar{y}_i \geq 0$ for $i = 1, 2, ..., m$.

It is easy to check that

\[
\bar{h}(\bar{y}_1, \bar{y}_2, ..., \bar{y}_{k-1}, \bar{y}_k, \bar{y}_{k+1}, ..., \bar{y}_m) = h(\bar{y}_1, \bar{y}_2, ..., \bar{y}_{k-1}, \bar{y}_k + \lambda, \bar{y}_{k+1}, ..., \bar{y}_m)
\]

and for any $(\bar{y}_1, \bar{y}_2, ..., \bar{y}_{k-1}, \bar{y}_k, \bar{y}_{k+1}, ..., \bar{y}_m)$ satisfying (0.5),

\[
(\bar{y}_1, \bar{y}_2, ..., \bar{y}_{k-1}, \bar{y}_k, \bar{y}_{k+1}, ..., \bar{y}_m) = (\bar{y}_1, \bar{y}_2, ..., \bar{y}_{k-1}, \bar{y}_k + \lambda, \bar{y}_{k+1}, ..., \bar{y}_m)
\]

obviously satisfies (0.4). Therefore, $\bar{h}_{\min} \geq h_{\min}$. 