A9.1 Find the residues of

(a) \( \frac{z}{z^4 + 1} \) at \( \exp(\pi i/4) \);

(b) \( \exp\left(z - \frac{1}{z}\right) \) at 0;

(c) \( \frac{z^2}{(z^2 - 1)^2} \) at 1;

(d) \( (\tan z)^3 \) at \( 3\pi/2 \).

Solution. (a) Since \( (z^4 + 1)' \neq 0 \) at \( z = \exp(\pi i/4) \), \( z(z^4 + 1)^{-1} \) has a simple pole at \( z = \exp(\pi i/4) \) with residue

\[
\text{Res}\left(\frac{z}{z^4 + 1}, \exp(\pi i/4)\right) = \left. \frac{z}{(z^4 + 1)'} \right|_{\exp(\pi i/4)} = -\frac{i}{4}
\]

(b) In \( 0 < |z| < \infty \),

\[
\exp\left(z - \frac{1}{z}\right) = \exp(z) \exp\left(-\frac{1}{z}\right) = \left(\sum_{l=0}^{\infty} \frac{z^l}{l!}\right) \left(\sum_{m=0}^{\infty} \frac{(-1)^m}{m!}z^{-m}\right) = \sum_{l,m \geq 0} \frac{(-1)^m}{l!m!} z^{l-m} = \sum_{n=-\infty}^{\infty} \left(\sum_{m=-n}^{\infty} \frac{(-1)^m}{(m+n)!m!}\right) z^n.
\]

Therefore,

\[
\text{Res}\left(\exp\left(z - \frac{1}{z}\right), 0\right) = \sum_{m=1}^{\infty} \frac{(-1)^m}{(m-1)!m!} = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{1!} - \frac{1}{2!} + \ldots
\]

(c) Since \( (z^2 - 1)^2 \) has a zero at 1 with multiplicity 2, \( z^2(z^2 - 1)^{-2} \) has a pole at 1 of order 2 and hence

\[
\text{Res}\left(\frac{z^2}{(z^2 - 1)^2}, 1\right) = \text{Res}\left(\frac{1}{(z - 1)^2 (z + 1)^2}, 1\right) = \left. \frac{z^2}{(z + 1)^2} \right|_{1} = \frac{1}{4}.
\]

\(^1\text{http://www.math.ualberta.ca/~xichen/math41117f/hw9s.pdf}\)
(d) Let \( z = w + 3\pi/2 \). In \( 0 < |w| < \pi \),

\[
(tan z)^3 = -(cot w)^3 = -(cos w)^3(sin w)^{-3}
\]

\[
= -w^{-3} \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} w^{2n} \right)^3 \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} w^{2n} \right)^{-3}
\]

Therefore, the residue of \((tan z)^3\) at \(3\pi/2\) is the coefficient of \(w^2\) in the expansion of the power series

\[
- \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} w^{2n} \right)^3 \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} w^{2n} \right)^{-3}
\]

\[
= - \left( 1 - \frac{w^2}{2} \right)^3 \left( 1 - \frac{w^2}{6} \right)^{-3} + O(w^4)
\]

\[
= - \left( 1 - \frac{3}{2}w^2 \right) \left( 1 + \frac{w^2}{2} \right) + O(w^4) = -1 + w^2 + O(w^4)
\]

where we use the notation \(O(w^m)\) to denote a power series in the form of

\[
O(w^m) = \sum_{n=m}^{\infty} a_n w^n.
\]

Therefore,

\[
Res \left( (tan z)^3, \frac{3\pi}{2} \right) = 1.
\]

A9.2 Compute the following integrals:

(a) \( \int_{-\pi}^{\pi} \frac{d\theta}{2 + \cos \theta + \sin \theta} \);

(b) \( \int_{0}^{\infty} \frac{\cos x}{x^4 + x^2 + 1} dx \).

Solution. (a) Let \( z = e^{i\theta} \). Then

\[
dz = ie^{i\theta} d\theta = iz d\theta, \quad \cos \theta = \frac{1}{2}(z + z^{-1}) \text{ and } \sin \theta = \frac{i}{2}(z^{-1} - z).
\]

Therefore,

\[
\frac{d\theta}{2 + \cos \theta + \sin \theta} = -\frac{iz^{-1}dz}{2 + \frac{1}{2}(z + z^{-1}) + \frac{i}{2}(z^{-1} - z)}
\]

\[
= \frac{1 - i}{z^2 + 2(1 + i)z + i} dz
\]
and
\[ \int_{-\pi}^{\pi} \frac{d\theta}{2 + \cos \theta + \sin \theta} = \int_{|z|=1} \frac{1 - i}{z^2 + 2(1 + i)z + i} \, dz \]

Note that \( z^2 + 2(1 + i)z + i \) has two roots
\[ \alpha_1 = \left(-1 + \frac{\sqrt{2}}{2}\right)(1 + i) \quad \text{and} \quad \alpha_2 = \left(-1 - \frac{\sqrt{2}}{2}\right)(1 + i) \]

with \(|\alpha_1| < 1\) and \(|\alpha_2| > 1\). Therefore,
\[ \int_{-\pi}^{\pi} \frac{d\theta}{2 + \cos \theta + \sin \theta} = \int_{|z|=1} \frac{1 - i}{z^2 + 2(1 + i)z + i} \, dz = 2\pi i \text{Res}_{\alpha_1}(z^2 + 2(1 + i)z + i) = 2\pi i \frac{1 - i}{\alpha_1 - \alpha_2} = \sqrt{2}\pi \]

(b) Since \( \cos x = \Re(e^{ix}) \),
\[ \int_{0}^{\infty} \frac{\cos x}{x^4 + x^2 + 1} \, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos x}{x^4 + x^2 + 1} \, dx \]
is the real part of
\[ \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{iz}}{z^4 + z^2 + 1} \, dz \]

For \( R > 0 \), let us consider the integral
\[ \int_{-R}^{R} \frac{e^{iz}}{z^4 + z^2 + 1} \, dz + \int_{C_R} \frac{e^{iz}}{z^4 + z^2 + 1} \, dz \]
where \( C_R \) is the semi-circle \( \{|z| = R, \text{Im}(z) \geq 0\} \).

For \( y = \text{Im}(z) \geq 0, |e^{iz}| = e^{-y} \leq 1 \). Therefore,
\[ \left| \int_{C_R} \frac{e^{iz}}{z^4 + z^2 + 1} \, dz \right| \leq \frac{\pi R}{R^4 - R^2 - 1} \]
and hence
\[ \int_{C_R} \frac{e^{iz}}{z^4 + z^2 + 1} \, dz = 0. \]

Note that
\[ z^4 + z^2 + 1 = (z^2 + 1)^2 - z^2 = (z^2 + z + 1)(z^2 - z + 1) = (z - e^{2\pi i/3})(z + e^{2\pi i/3})(z - e^{\pi i/3})(z + e^{\pi i/3}) \]
where \( \text{Im}(e^{2\pi i/3}) > 0 \) and \( \text{Im}(e^{\pi i/3}) > 0 \). Hence
\[
\int_{-R}^{R} \frac{e^{iz}}{z^4 + z^2 + 1} \, dz + \int_{C_R} \frac{e^{iz}}{z^4 + z^2 + 1} \, dz \\
= 2\pi i \text{Res} \left( \frac{e^{iz}}{z^4 + z^2 + 1}, e^{2\pi i/3} \right) + 2\pi i \text{Res} \left( \frac{e^{iz}}{z^4 + z^2 + 1}, e^{\pi i/3} \right) \\
= 2\pi i \left( \frac{e^{iz}}{(z^4 + z^2 + 1)'} \right) \bigg|_{z = e^{2\pi i/3}} + 2\pi i \left( \frac{e^{iz}}{(z^4 + z^2 + 1)'} \right) \bigg|_{z = e^{\pi i/3}} \\
= \frac{\pi}{3} e^{-\sqrt{3}/2} \left( \sqrt{3} \cos \left( \frac{1}{2} \right) + 3 \sin \left( \frac{1}{2} \right) \right).
\]

Taking the limit as \( R \to \infty \), we obtain
\[
\int_{0}^{\infty} \frac{\cos x}{x^4 + x^2 + 1} \, dx = \lim_{R \to \infty} \int_{-R}^{R} \frac{e^{iz}}{z^4 + z^2 + 1} \, dz \\
= \frac{\pi}{6} e^{-\sqrt{3}/2} \left( \sqrt{3} \cos \left( \frac{1}{2} \right) + 3 \sin \left( \frac{1}{2} \right) \right).
\]

□

A9.3 Let \( f(z) \) be a nonconstant analytic function on \( \mathbb{C} \setminus S \), where \( S \) is a finite subset of \( \mathbb{C} \). Show that \( f(\mathbb{C} \setminus S) = \mathbb{C} \).

Proof. If \( f(z) \) has an essential singularity at some \( p \in S \), then by Casorati-Weierstrass Theorem,
\[
\overline{f(\mathbb{C} \setminus S)} \supset \overline{f(\{0 < |z - p| < r\})} = \mathbb{C}
\]
for all \( r > 0 \) such that \( \{0 < |z - p| < r\} \cap S = \emptyset \).

If \( f(z) \) has an essential singularity at \( \infty \), then by Casorati-Weierstrass Theorem again,
\[
\overline{f(\mathbb{C} \setminus S)} \supset \overline{f(\{|z| > R\})} = \mathbb{C}
\]
for all \( R > 0 \) such that \( S \subset \{|z| < R\} \).

Suppose that \( f(z) \) has at worst poles at \( S \) and \( \infty \). Let \( S = \{z_1, z_2, \ldots, z_n\} \) and \( m_j \) be the order of the pole of \( f(z) \) at \( z_j \) for \( j = 1, 2, \ldots, m \). We let
\[
g(z) = (z - z_1)^{m_1}(z - z_2)^{m_2} \ldots (z - z_n)^{m_n} \text{ and } h(z) = f(z)g(z).
\]

Then \( h(z) \) has removable singularities at \( S \) and is hence entire. Since \( f(z) \) and \( g(z) \) have at worst poles at \( \infty \), the same holds for \( h(z) = f(z)g(z) \). In conclusion, \( h(z) \) is an entire function with at worst a pole at \( \infty \). Therefore, \( h(z) \) is a polynomial and \( f(z) = h(z)/g(z) \) is a rational function in \( z \).
Let

\[ M = \lim_{z \to \infty} \frac{h(z)}{g(z)} \]

where we set \( M = \infty \) if \( \deg h(z) > \deg g(z) \). For all \( c \neq M \), \( h(z) - cg(z) \) is a nonzero polynomial. And since \( f(z) \) is not constant, \( h(z) - cg(z) \) is a polynomial of degree at least one. By Fundamental Theorem of Algebra, there exists \( z_0 \) such that \( h(z_0) - cg(z_0) = 0 \) and hence \( c \in f(\mathbb{C} \setminus S) \). Consequently,

\[ \overline{f(\mathbb{C} \setminus S)} \supset \mathbb{C} \setminus \{M\} = \mathbb{C}. \]

□

A9.4 Let \( f : \mathbb{C}^* \to \mathbb{C}^* \) be a biholomorphic map. Do the following:

(a) Show that \( f(z) \) has removable singularities or poles at both 0 and \( \infty \).

(b) Show that \( f(z) \equiv cz \) or \( cz^{-1} \) for some constant \( c \neq 0 \).

Proof. Let \( D = \{|z - 2| < 1\} \). By Open Mapping Theorem, \( f(D) \) contains a disk at \( f(2) \), i.e.,

\[ \{|w - f(2)| < r\} \subset f(D) \]

for some \( r > 0 \). Since \( f \) is one-to-one,

\[ f(D) \cap f(\mathbb{C} \setminus D) = \emptyset \]

and hence

\[ \{|w - f(2)| < r\} \cap f(\mathbb{C} \setminus D) = \emptyset. \]

Therefore,

\[ f(2) \notin \overline{f(\mathbb{C} \setminus D)}. \]

If \( f(z) \) has an essential singularity at 0, then by Casorati-Weierstrass,

\[ \overline{f(\mathbb{C} \setminus D)} \supset f(\{0 < |z| < 1\}) = \mathbb{C} \]

which is a contraction.

If \( f(z) \) has an essential singularity at \( \infty \), then by Casorati-Weierstrass,

\[ \overline{f(\mathbb{C} \setminus D)} \supset f(\{|z| > 3\}) = \mathbb{C} \]

which is again a contraction.

Consequently, \( f(z) \) has at worst poles at 0 and \( \infty \). Suppose that

\[ f(z) = \sum_{n \geq 0} a_n z^n + \sum_{n < 0} a_n z^n \]
is the Laurent series of \( f(z) \) in \( 0 < |z| < \infty \). Since \( f(z) \) has at worst a pole at 0, all but finitely many \( a_n \) vanish for \( n < 0 \). Since \( f(z) \) has at worst a pole at \( \infty \), all but finitely many \( a_n \) vanish for \( n > 0 \). Therefore,

\[
f(z) = \sum_{n=-m}^{m} a_n z^n = \frac{g(z)}{z^m}
\]

for some \( m \in \mathbb{N} \) and some polynomial \( g(z) \) such that \( g(z) \) and \( z^m \) are coprime.

We discuss in three separate cases.

Suppose that \( g(z) \) is constant. Since \( f(z) \) is not constant, \( f(z) = cz^{-m} \) for some \( c \neq 0 \) and \( m > 0 \). If \( m > 1 \), then

\[
f(e^{2\pi i/m}) = f(1)
\]

and hence \( f \) is not one-to-one. So \( m = 1 \) and \( f(z) = cz^{-1} \).

Suppose that \( m = 0 \). Then \( f(z) = g(z) \) is a polynomial of degree at least one. Since \( f \) is biholomorphic on \( \mathbb{C}^* \), \( f'(z) \neq 0 \) for all \( z \neq 0 \). So \( f'(z) = 0 \) only if \( z = 0 \). Hence \( f'(z) = az^{n-1} \) and \( f(z) = cz^n + b \) for some constants \( c \neq 0 \) and \( b \in \mathbb{C} \). If \( n > 1 \), then

\[
f(e^{2\pi i/n}) = f(1)
\]

and hence \( f \) is not one-to-one. So \( n = 1 \) and \( f(z) = cz + b \). If \( b \neq 0 \), then \( f(-b/c) = 0 \). Therefore, \( b = 0 \) and \( f(z) = cz \).

Suppose that \( m > 0 \) and \( g(z) \) is not constant. Then \( g(0) \neq 0 \) since \( g(z) \) and \( z^m \) are coprime. By Fundamental Theorem of Algebra, \( zg'(z) - mg(z) \) has a root \( z_0 \). Since \( g(0) \neq 0 \), \( z_0 \neq 0 \). So

\[
f'(z_0) = \frac{z_0 g'(z_0) - mg(z_0)}{z_0^{m+1}} = 0
\]

which contradicts the fact that \( f'(z) \neq 0 \) for all \( z \neq 0 \).

In conclusion, \( f(z) = cz \) or \( cz^{-1} \) for some constant \( c \neq 0 \). □

A9.5 Do the following:

(a) Compute

\[
\sum_{n=1}^{\infty} \frac{1}{n^4}
\]

by considering the complex line integral

\[
\int_{\gamma_n} \frac{dz}{z^4 \sin z}
\]
along the boundary $\gamma_n$ of the rectangle
\[
\left\{ |x|, |y| \leq n\pi + \frac{\pi}{2} \right\}.
\]

(b) (Bonus +20 points) Show that there exists a sequence of positive rational numbers $c_m$ such that
\[
\sum_{n=1}^{\infty} \frac{1}{n^{2m}} = c_m\pi^{2m}
\]
for $m = 1, 2, \ldots$

Proof. For all $m \in \mathbb{Z}^+$, we consider the integral
\[
\int_{\gamma_n} \frac{dz}{z^{2m} \sin z}
\]
When $x = \pm(n\pi + \pi/2)$,
\[
|\sin z|^2 = |\sin x|^2 + \left( \frac{e^y - e^{-y}}{2} \right)^2 \geq \sin^2 \left( n\pi + \frac{\pi}{2} \right) = 1.
\]
When $y = \pm(n\pi + \pi/2)$,
\[
|\sin z|^2 = |\sin x|^2 + \left( \frac{e^y - e^{-y}}{2} \right)^2 \geq \left( \frac{e^{n\pi+\pi/2} - e^{-n\pi-\pi/2}}{2} \right)^2 > 1.
\]
Therefore,
\[
\left| \int_{\gamma_n} \frac{dz}{z^{2m} \sin z} \right| \leq 8 \left( n\pi + \frac{\pi}{2} \right) \left( n\pi + \frac{\pi}{2} \right)^{-2m}
\]
and hence
\[
\lim_{n \to \infty} \int_{\gamma_n} \frac{dz}{z^{2m} \sin z} = 0
\]
for $m \geq 1$. By Cauchy Integral Theorem,
\[
\frac{1}{2\pi i} \int_{\gamma_n} \frac{dz}{z^{2m} \sin z} = \sum_{l=-n}^{n} \text{Res}(z^{-2m}(\sin z)^{-1}, l\pi)
\]
For $l \neq 0$, $z^{-2m}(\sin z)^{-1}$ has a simple pole at $l\pi$ and
\[
\text{Res}(z^{-2m}(\sin z)^{-1}, l\pi) = \frac{1}{z^{2m}(\sin z)^{\prime}} \bigg|_{z=l\pi} = \frac{(-1)^l}{l^{2m}\pi^{2m}}
\]
Therefore,
\[
\frac{1}{2\pi i} \int_{\gamma_n} \frac{dz}{z^{2m} \sin z} = \text{Res}(z^{-2m}(\sin z)^{-1}, 0) + 2 \sum_{l=1}^{n} \frac{(-1)^l}{l^{2m}\pi^{2m}}
\]
Taking limits as $n \to \infty$, we obtain
\[
2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{2m}} = \pi^{2m} \text{Res}(z^{-2m}(\sin z)^{-1}, 0)
\]
where
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{2m}} = \sum_{n=1}^{\infty} \frac{1}{n^{2m}} - 2 \sum_{n=1}^{\infty} \frac{1}{(2n)^{2m}} = (1 - 2^{1-2m}) \sum_{n=1}^{\infty} \frac{1}{n^{2m}}.
\]
Therefore,
\[
\sum_{n=1}^{\infty} \frac{1}{n^{2m}} = \frac{\pi^{2m}}{2 - 2^{2-2m}} \text{Res}(z^{-2m}(\sin z)^{-1}, 0)
\]
In $0 < |z| < \pi$,
\[
z^{-2m}(\sin z)^{-1} = z^{-2m-1} \left(1 - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k + 1)!} z^{2k}\right)^{-1}
\]
\[
= z^{-2m-1} \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k + 1)!} z^{2k}\right)^n
\]
and hence $\text{Res}(z^{-2m}(\sin z)^{-1}, 0)$ is the coefficient of $z^{2m}$ in the expansion of
\[
\sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k + 1)!} z^{2k}\right)^n
\]
which is the same as the coefficient of $z^{2m}$ in the rational polynomial
\[
\sum_{n=1}^{m} \left(\sum_{k=1}^{m+n} \frac{(-1)^{k+1}}{(2k + 1)!} z^{2k}\right)^n
\]
Therefore, $\text{Res}(z^{-2m}(\sin z)^{-1}, 0)$ is a rational number and
\[
\sum_{n=1}^{\infty} \frac{1}{n^{2m}} = c_m \pi^{2m}
\]
for some $c_m \in \mathbb{Q}^+$.

When $m = 2$, $\text{Res}(z^{-4}(\sin z)^{-1}, 0)$ is the coefficient of $z^4$ in the expansion of
\[
\sum_{n=1}^{2} \left(\sum_{k=1}^{3-n} \frac{(-1)^{k+1}}{(2k + 1)!} z^{2k}\right)^n = \left(\frac{z^2}{6} - \frac{z^4}{120}\right) + \left(\frac{z^2}{6}\right)^2
\]
\[
= \frac{z^2}{6} + \frac{7}{360} z^4
\]
Therefore, \( \text{Res}(z^{-4}(\sin z)^{-1}, 0) = \frac{7}{360} \) and

\[
\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{2 - 2^{-2}} \text{ Res}(z^{-4}(\sin z)^{-1}, 0) = \frac{\pi^4}{90}
\]