A7.1 Show that each of the following sequences of entire functions $f_n$ converges compactly on $\mathbb{C}$:

(a) $f_n(z) = \sum_{m=1}^{n} \frac{z^m}{m^m}$

(b) $f_n(z) = \sum_{m=1}^{n} \frac{1}{m^2} \exp\left(\frac{z}{m}\right)$

Proof. (a) This sequence of the partial sums of the power series $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^n}$ whose radius of convergence is

$$\left(\lim_{n \to \infty} \sqrt[n]{\frac{1}{n^n}}\right)^{-1} = \infty.$$  

So for all $R > 0$, $f_n$ converges uniformly in $\{|z| \leq R\}$.

(b) It suffices to prove that for every $R > 0$, there exists a sequence $c_n > 0$ such that

$$\left|\frac{1}{n^2} \exp\left(\frac{z}{n}\right)\right| \leq c_n \quad \text{and} \quad \sum_{n=1}^{\infty} c_n < \infty$$

in $\{|z| \leq R\}$. When $|z| \leq R$, $|z/n| \leq R/n \leq R$ for all positive integer $n$.

So

$$\left|\exp\left(\frac{z}{n}\right)\right| \leq M = \max_{|z| \leq R} |\exp(z)| \Rightarrow \left|\frac{1}{n^2} \exp\left(\frac{z}{n}\right)\right| \leq \frac{M}{n^2}.$$  

And since

$$\sum_{m=1}^{n} \frac{M}{m^2} \leq \frac{M}{1^2} + \int_{1}^{n-1} \frac{M}{x^2} dx = 2M - \frac{M}{n-1},$$

we conclude that

$$\sum_{n=1}^{\infty} \frac{M}{n^2}$$

converges. Therefore, $f_n(z)$ converges uniformly in $\{|z| \leq R\}$ for every $R > 0$.  \qed
A7.2 Show that for all complex numbers \( z_1, z_2, z_3, w_1, w_2, w_3 \) satisfying \( z_1 \neq z_2 \neq z_3 \) and \( w_1 \neq w_2 \neq w_3 \), there exists a unique linear fraction transformation

\[
f(z) = \frac{az + b}{cz + d}
\]

such that \( f(z_j) = w_j \) for \( j = 1, 2, 3 \).

Proof. To find \( f(z_j) = w_j \), we need to solve the system of linear equations

\[
\begin{align*}
az_1 + b - cz_1 w_1 - dw_1 &= 0 \\
az_2 + b - cz_2 w_2 - dw_2 &= 0 \\
az_3 + b - cz_3 w_3 - dw_3 &= 0
\end{align*}
\]

or in matrix notation

\[
\begin{bmatrix}
z_1 & 1 & -z_1 w_1 & -w_1 \\
z_2 & 1 & -z_2 w_2 & -w_2 \\
z_3 & 1 & -z_3 w_3 & -w_3 \\
\end{bmatrix} \begin{bmatrix}
a \\
b \\
c \\
d
\end{bmatrix} = 0
\]

For every nonzero solution \((a, b, c, d)\) of the above system of linear equations, \( f(z_j) = w_j \); and since \( f(z) \) is not constant, we necessarily have \( ad - bc \neq 0 \) and it is hence a linear fraction transformation. Two nonzero solutions \((a, b, c, d)\) and \((a', b', c', d')\) give the same linear fractional transformation if and only if they span the same vector space, i.e.,

\[
\text{Span}\{(a, b, c, d)\} = \text{Span}\{(a', b', c', d')\}.
\]

So there exists a unique linear fractional transformation \( f(z) \) sending \( f(z_j) = w_j \) if and only if the solutions of the above system of linear equations are a one-dimensional vector subspace of \( \mathbb{C}^4 \), i.e., \( \text{rank}(A) = 3 \). It remains to prove that \( \text{rank}(A) = 3 \).
Applying column operations to $A$, we have

$$\text{rank}(A) = \text{rank} \begin{bmatrix} z_1 & 1 & -z_1 w_1 & -w_1 \\ z_2 & 1 & -z_2 w_2 & -w_2 \\ z_3 & 1 & -z_3 w_3 & -w_3 \end{bmatrix}$$

$$= \text{rank} \begin{bmatrix} z_1 & 1 & 0 & -w_1 \\ z_2 & 1 & z_2 (w_1 - w_2) & -w_2 \\ z_3 & 1 & z_3 (w_1 - w_3) & -w_3 \end{bmatrix}$$

$$= \text{rank} \begin{bmatrix} 0 & 1 & 0 & 0 \\ z_1 - z_2 & 1 & z_2 (w_1 - w_2) & w_1 - w_2 \\ z_1 - z_3 & 1 & z_3 (w_1 - w_3) & w_1 - w_3 \end{bmatrix}$$

$$= 1 + \text{rank} \begin{bmatrix} z_1 - z_2 & z_2 (w_1 - w_2) & 0 & w_1 - w_2 \\ z_1 - z_3 & z_3 (w_1 - w_3) & 0 & w_1 - w_3 \end{bmatrix}$$

$$= 2 + \text{rank} \begin{bmatrix} z_1 - z_2 & w_1 - w_2 \end{bmatrix} = 3$$

since $z_1 \neq z_2 \neq z_3$ and $w_1 \neq w_2 \neq w_3$.

$$\square$$

A7.3 Let $f(z)$ be an analytic function in $D = \{ |z| < 1 \}$. Show that if $|f(z)| < 1$ for all $z \in D$ and $f^{(k)}(0) = 0$ for $k = 0, 1, \ldots, m-1$, then $|f(z)| \leq |z|^m$ for all $z \in D$ and $|f^{(m)}(0)| \leq m!$.

**Proof.** Since $f^{(k)}(0) = 0$ for $k = 0, 1, \ldots, m-1$,

$$f(z) = \sum_{n=m}^{\infty} a_n z^n = z^m \sum_{n=m} a_n z^{n-m}$$

where $a_m = f^{(m)}(0)/m!$. We let

$$g(z) = \sum_{n=m} a_n z^{n-m}.$$ 

Then $g(z) = f(z)/z^m$ for $0 < |z| < 1$. For all $0 < r < 1$,

$$\max_{|z| \leq r} |g(z)| \leq \max_{|z|=r} |g(z)| \leq \max_{|z|=r} \left| \frac{f(z)}{z^m} \right|$$

$$= \frac{1}{r^m} \max_{|z|=r} |f(z)| \leq \frac{1}{r^m}$$

Taking $r \to 1$, we obtain

$$\sup_{|z|<1} |g(z)| \leq 1.$$
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That is, $|g(z)| \leq 1$ for all $|z| < 1$. Therefore,

$$|f(z)| = |z^m g(z)| = |z|^m |g(z)| \leq |z|^m$$

for all $|z| < 1$. At $z = 0$, we have

$$1 \leq |g(0)| = |a_m| = \frac{|f^{(m)}(0)|}{m!} \Rightarrow |f^{(m)}(0)| \leq m!$$

A7.4 Let $f(z)$ be an analytic function in $D = \{|z - z_0| < R\}$. Show that

$$\sup_{z \in D} |f(z) - f(z_0)| \geq R |f'(z_0)|.$$

Proof. Let

$$M = \sup_{z \in D} |f(z) - f(z_0)|.$$

If $M = \infty$, there is nothing to prove. Otherwise, $M < \infty$ and we let

$$g(z) = \frac{f(z_0 + Rz) - f(z_0)}{M}$$

Then $g(z)$ is an analytic function on $\{|z| < 1\}$. For $|z| < 1$, $z_0 + Rz \in D$ and hence

$$|f(z_0 + Rz) - f(z_0)| \leq M \Rightarrow |g(z)| = 1.$$

Also it is obvious that $g(0) = 0$. So by Schwartz Lemma, $|g'(0)| \leq 1$. That is,

$$1 \geq |g'(0)| = \frac{R |f'(z_0)|}{M} \Rightarrow M \geq R |f'(z_0)|.$$

□

A7.5 Show that if two entire functions $f(z)$ and $g(z)$ satisfy that $|f(z)| \leq |g(z)|$ for all $z$, then $f(z) \equiv cg(z)$ for a constant $c$ with $|c| \leq 1$.

Proof. By A6.1, there exists an entire function $h(z)$ such that $f(z) \equiv h(z)g(z)$.

If $g(z) \equiv 0$, then $f(z) \equiv 0$ and we simply let $c = 1$.

Suppose that $g(z) \not\equiv 0$. Then $|h(z)| \leq 1$ for $g(z) \not\equiv 0$. By continuity, $|h(z)| \leq 1$ for all $z$. So by Louville, $h(z) \equiv c$ with $|c| \leq 1$. □