A6.1 Let \( f(z) \) and \( g(z) \) be two analytic functions on an open set \( D \subset \mathbb{C} \) satisfying that \(|f(z)| \leq |g(z)|\) for all \( z \in D \). Show that there exists an analytic function \( h(z) \) on \( D \) such that

\[
f(z) = h(z)g(z).
\]

**Proof.** It suffices to find \( h(z) \) on each connected component of \( D \). So let us assume that \( D \) is connected.

If \( f(z) \equiv 0 \), then we simply take \( h(z) \equiv 0 \). Suppose that \( f(z) \not\equiv 0 \). Since \(|f(z)| \leq |g(z)|\), \( g(z) \not\equiv 0 \).

Suppose that \( g(z_0) = 0 \). By power series expansion of analytic functions, we can write

\[
f(z) = (z - z_0)^m p(z) \quad \text{and} \quad g(z) = (z - z_0)^n q(z)
\]

in \( \{|z - z_0| < r\} \) for some \( r > 0 \) such that \( p(z) \) and \( q(z) \) are analytic functions in \( \{|z - z_0| < r\} \) satisfying \( p(z_0) \neq 0 \) and \( q(z_0) \neq 0 \).

We claim that \( m \geq n \). Otherwise, if \( m < n \), then

\[
\lim_{z \to z_0} \frac{g(z)}{f(z)} = \lim_{z \to z_0} (z - z_0)^{n-m} \left( \frac{q(z)}{p(z)} \right) = 0
\]

which contradicts the fact that

\[
\left| \frac{g(z)}{f(z)} \right| \geq 1
\]

in \( \{0 < |z - z_0| < r\} \).

So \( m \geq n \). Since \( g^{(n)}(z_0) = (n!)q(z_0) \), we have

\[
(z - z_0)^{m-n} \left( \frac{p(z)}{q(z)} \right) \bigg|_{z_0} = \frac{f^{(n)}(z_0)}{g^{(n)}(z_0)}
\]

Therefore, we can define \( h(z) \) as

\[
h(z) = \begin{cases} 
  f(z) / g(z) & \text{if } g(z) \neq 0 \\
  f^{(n)}(z_0) / g^{(n)}(z_0) & \text{at } z_0 \text{ with } g(z_0) = 0
\end{cases}
\]

where \( n \) is the multiplicity of \( g(z) \) at \( z_0 \). To show that \( h(z) \) is analytic in \( D \), it suffices to show that it is analytic at every point.
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\( z_0 \in D \). If \( g(z_0) \neq 0 \), \( h(z) = f(z)/g(z) \) is obviously analytic at \( z_0 \). If \( g(z_0) = 0 \), then
\[
    h(z) \equiv (z - z_0)^{m-n} \left( \frac{p(z)}{q(z)} \right)
\]
in \( \{|z - z_0| < r\} \) and hence it is also analytic at \( z_0 \).  \( \square \)

A6.2 Let \( f(z) = u(x, y) + iv(x, y) \) be an analytic function on a connected open set \( D \subset \mathbb{C} \). Show that if the real function \( u(x, y) - v(x, y) \) has a local extreme in \( D \), then \( f(z) \) must be constant.

Proof. We can prove the following more general statement: Let \( h : \mathbb{R}^2 \rightarrow \mathbb{R} \) be a function that does not have any local extreme. If \( h(u(x, y), v(x, y)) \) has a local extreme in \( D \), then \( f(z) \) must be constant.

Suppose that \( f(z) \) is not constant and \( h(u(x, y), v(x, y)) \) has a local extreme at \( (x_0, y_0) \). Using complex variable, we write \( h(f(z)) = h(u(x, y), v(x, y)) \). Then there exists \( R > 0 \) such that
\[
    h(f(z)) \geq h(f(z_0)) \quad (h(f(z)) \leq h(f(z_0))) \quad \text{for all } |z - z_0| < R
\]
By open mapping theorem, \( f(\{|z - z_0| < R\}) \) contains a disk at \( f(z_0) \). That is,
\[
    \{|w - f(z_0)| < r\} \subset f(\{|z - z_0| < R\})
\]
for some \( r > 0 \). Then
\[
    h(w) \geq h(w_0) \quad (h(w) \leq h(w_0)) \quad \text{for all } |w - w_0| < r
\]
where \( w_0 = f(z_0) \). This means \( h(w) \) has a local extreme at \( w_0 \), contradicting to our hypothesis.

This proves our claim. To apply it to our situation, let \( h(x, y) = x - y \). It has no local extreme since \( h_x \equiv 1 \neq 0 \). If \( u(x, y) - v(x, y) = h(u(x, y), v(x, y)) \) has a local extreme in \( D \), then \( f(z) \) must be constant.  \( \square \)

A6.3 Let \( f(z) \) be an entire function. Show that if
\[
    \lim_{z \to \infty} \frac{|f(z)|}{|z|^r} < \infty
\]
for some real number \( r \geq 0 \), then \( f(z) \) must be a polynomial of degree \( \deg f(z) \leq r \).
Proof. By Generalized Cauchy Integral Formula,

\[ f^{(n)}(0) = \frac{n!}{2\pi i} \int_{|z|=R} \frac{f(z)}{z^{n+1}} \, dz \]

for all \( n \in \mathbb{N} \) and \( R > 0 \). Then we have

\[ |f^{(n)}(0)| \leq \frac{n!}{2\pi} (2\pi R) \frac{1}{R^{n+1}} \max_{|z|=R} |f(z)| = \frac{n!}{R^n} \max_{|z|=R} |f(z)| \]

Taking \( R \to \infty \), we have

\[ |f^{(n)}(0)| \leq (n!) \lim_{z \to \infty} \frac{|f(z)|}{|z|^n} \]

For \( n > r \),

\[ \lim_{z \to \infty} \frac{|f(z)|}{|z|^n} = \left( \lim_{z \to \infty} \frac{|f(z)|}{|z|^r} \right) \lim_{z \to \infty} \frac{1}{|z|^{n-r}} = 0 \]

Therefore, \( f^{(n)}(0) = 0 \) for all \( n > r \). That is,

\[ f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n \leq r} \frac{f^{(n)}(0)}{n!} z^n \]

is a polynomial of degree at most \( r \). \( \square \)

A6.4 Let \( f(z) \) be an entire function satisfying

\[ |f(z_1 + z_2)| \leq |f(z_1) + f(z_2)| \]

for all complex numbers \( z_1 \) and \( z_2 \). Show that \( f(z) \) is a polynomial of degree at most 1.

Proof. We claim that

\[ |f(z_1 + z_2 + \ldots + z_n)| \leq |f(z_1)| + |f(z_2)| + \ldots + |f(z_n)| \]

for all \( z_1, z_2, \ldots, z_n \in \mathbb{C} \). We prove this by induction.

It is trivial for \( n = 1 \). Suppose that

\[ |f(z_1 + z_2 + \ldots + z_{n-1})| \leq |f(z_1)| + |f(z_2)| + \ldots + |f(z_{n-1})|. \]

Then

\[ |f(z_1 + z_2 + \ldots + z_n)| = |f((z_1 + z_2 + \ldots + z_{n-1}) + z_n)| \]

\[ \leq |f(z_1 + z_2 + \ldots + z_{n-1}) + f(z_n)| \]

\[ \leq |f(z_1 + z_2 + \ldots + z_{n-1})| + |f(z_n)| \]

\[ \leq |f(z_1)| + |f(z_2)| + \ldots + |f(z_{n-1})| + |f(z_n)|. \]

This proves our claim.
Let \( z_1 = z_2 = \ldots z_n = z/n \). Then
\[
|f(z)| \leq n \left| f\left(\frac{z}{n}\right) \right|.
\]
For every complex number \( z \), we let \( n \) be a positive integer such that \( n - 1 \leq |z| \leq n \). Then \( |z/n| \leq 1 \) and
\[
|f(z)| \leq n \left| f\left(\frac{z}{n}\right) \right| \leq nM \leq M(|z| + 1)
\]
where \( M \) is the maximum of \( |f(z)| \) on \( \{|z| \leq 1\} \). Therefore,
\[
\lim_{z \to \infty} \frac{|f(z)|}{|z|} < \infty
\]
By A6.3, \( f(z) \) is a polynomial of degree at most 1. \( \square \)

A6.5 Let \( f(z) \) be an entire function with two periods \( \lambda_1 \) and \( \lambda_2 \), i.e.,
\[
f(z) = f(z + \lambda_1) = f(z + \lambda_2)
\]
for all \( z \in \mathbb{C} \). Show that if \( \lambda_1 \) and \( \lambda_2 \) are linearly independent over \( \mathbb{R} \), then \( f(z) \) must be constant.
Bonus (+10 points): If we only assume that \( \lambda_1 \) and \( \lambda_2 \) are linearly independent over \( \mathbb{Q} \), does \( f(z) \) have to be constant?

Proof. Suppose that \( \lambda_1 \) and \( \lambda_2 \) are linearly independent over \( \mathbb{R} \). Then every complex number \( z \) is a linear combination of \( \lambda_1 \) and \( \lambda_2 \) over \( \mathbb{R} \). That is,
\[
z = c_1 \lambda_1 + c_2 \lambda_2
\]
for some real numbers \( c_1 \) and \( c_2 \). Let
\[
m_1 = \lfloor c_1 \rfloor \text{ and } m_2 = \lfloor c_2 \rfloor
\]
be the largest integers less than or equal to \( c_1 \) and \( c_2 \), respectively. Then
\[
f(z) = f(z - m_1 \lambda_1 - m_2 \lambda_2) = f((c_1 - m_1)\lambda_1 + (c_2 - m_2)\lambda_2).
\]
Since \( 0 < c_1 - m_1 \leq 1 \) and \( 0 < c_2 - m_2 \leq 1 \),
\[
|(c_1 - m_1)\lambda_1 + (c_2 - m_2)\lambda_2| \leq (c_1 - m_1)|\lambda_1| + (c_2 - m_2)|\lambda_2| \leq |\lambda_1| + |\lambda_2|.
\]
Let \( M \) be the maximum of \( |f(z)| \) on \( \{|z| \leq |\lambda_1| + |\lambda_2|\} \). Then
\[
|f(z)| = |f((c_1 - m_1)\lambda_1 + (c_2 - m_2)\lambda_2)| \leq M
\]
for all \( z \in \mathbb{C} \). So \( f(z) \) is constant by Louville.

Suppose that \( \lambda_1 \) and \( \lambda_2 \) are linearly independent over \( \mathbb{Q} \). If they are linearly independent over \( \mathbb{R} \), then we are done. Otherwise, \( \lambda_1 \) and \( \lambda_2 \) are linearly independent over \( \mathbb{Q} \) and dependent
over $\mathbb{R}$. That is, $\lambda = \lambda_1/\lambda_2 \in \mathbb{R}\setminus\mathbb{Q}$. Namely, it is an irrational real number.

We claim that for every $\varepsilon > 0$, there exist integers $m_1$ and $m_2$ such that
\[0 < |m_1 \lambda - m_2| < \varepsilon\]
Fixing a positive integer $n$, let us consider
\[a_k = k\lambda - \lfloor k\lambda \rfloor\]
for $k = 0, 1, 2, \ldots, n$. These are $n + 1$ numbers in the interval $[0, 1]$. By Pigeon Hole principle, there exist $a_k$ and $a_l$ such that $0 \leq k \neq l \leq n$ and
\[|a_k - a_l| = |(k - l)\lambda - (\lfloor k\lambda \rfloor - \lfloor l\lambda \rfloor)| \leq \frac{1}{n}\]
Let $m_1 = k - l$ and $m_2 = \lfloor k\lambda \rfloor - \lfloor l\lambda \rfloor$. Then
\[|m_1 \lambda - m_2| \leq \frac{1}{n}\]
This proves our claim.

For every positive integer $n$, there exist integers $m_1$ and $m_2$ such that
\[0 < |m_1 \lambda - m_2| \leq \frac{1}{n}\]
So
\[0 < |m_1 \lambda_1 - m_2 \lambda_2| = |\lambda_2(m_1 \lambda - m_2)| \leq \frac{\lambda_2}{n}\]
Let $z_n = m_1 \lambda_1 - m_2 \lambda_2$. Since $f(z_n) = f(m_1 \lambda_1 - m_2 \lambda_2) = f(0)$, we conclude that there exists a sequence $\{z_n\}$ such that
\[0 < |z_n| \leq \frac{\lambda_2}{n} \quad \text{and} \quad f(z_n) = f(0)\]
This means that the set $\{z : f(z) = f(0)\}$ has a cluster point at $0$. So $f(z) \equiv f(0)$. \qed