A5.1 Compute the following integrals. You must justify your answer.

(a) $\int \gamma (z^2 + e^z) \, dz$, where $\gamma(t) = t + t^2$ for $t \in [0, 1]$.

(b) $\int \frac{dz}{z^2(z - 2)}$, where $\gamma$ is the boundary of the square $\{ |x|, |y| \leq 1 \}$ oriented counterclockwise.

(c) $\int_{|z|=2} \frac{dz}{z^{2017} + z^{2016} + 1}$, where the circle $\{ |z| = 2 \}$ is oriented counterclockwise.

(d) $\int_{|z|=3} \frac{e^z}{z^2 + 1} \, dz$, where the circle $\{ |z| = 3 \}$ is oriented counterclockwise.

Solution. (a) Since $(\frac{z^3}{3} + e^z)' = z^2 + e^z$,

$$\int \gamma (z^2 + e^z) \, dz = \left( \frac{z^3}{3} + e^z \right) \bigg|_{\gamma(0)}^{(1)} = e^2 + \frac{5}{3}.$$ 

(b) By CIT and CIF,

$$\int \frac{dz}{z^2(z - 2)} = \int_{|z|=1/2} \frac{dz}{z^2(z - 2)} = \frac{2\pi i}{1!} \left( \frac{1}{z - 2} \right) \bigg|_{z=0} = -\frac{\pi i}{2}.$$ 

(c) We claim that $z^{2017} + z^{2016} + 1$ has no zeros in $\{ |z| \geq 2 \}$. Otherwise, suppose that $z^{2017} + z^{2016} + 1 = 0$ for some $|z| \geq 2$. Then

$$0 = \left| \frac{z^{2017} + z^{2016} + 1}{z^{2017}} \right| = \left| 1 + \frac{1}{z} + \frac{1}{z^{2017}} \right| \geq 1 - \frac{1}{|z|} - \frac{1}{|z|^{2017}} \geq 1 - \frac{1}{2} - \frac{1}{2^{2017}} > 0$$

which is a contradiction. Therefore, $(z^{2017} + z^{2016} + 1)^{-1}$ is analytic on $\{ |z| \geq 2 \}$. By CIT,

$$\int_{|z|=2} \frac{dz}{z^{2017} + z^{2016} + 1} = \int_{|z|=R} \frac{dz}{z^{2017} + z^{2016} + 1}.$$ 

\[ \text{http://www.math.ualberta.ca/~xichen/math41117f/hw5sol.pdf} \]
for all $R \geq 2$. Then

$$\left| \int_{|z|=2} \frac{dz}{z^{2017} + z^{2016} + 1} \right| = \left| \int_{|z|=R} \frac{dz}{z^{2017} + z^{2016} + 1} \right| \leq \frac{2\pi R}{R^{2017} - R^{2016} - 1}$$

for all $R \geq 2$. When $R \to \infty$, we obtain

$$\left| \int_{|z|=2} \frac{dz}{z^{2017} + z^{2016} + 1} \right| = 0 \Rightarrow \int_{|z|=2} \frac{dz}{z^{2017} + z^{2016} + 1} = 0.$$  

(d) By CIT and CIF,

$$\int_{|z|=3} \frac{e^z}{z^2 + 1} dz = \int_{|z-i|=1} \frac{e^z}{z^2 + 1} dz + \int_{|z+i|=1} \frac{e^z}{z^2 + 1} dz = 2\pi i \left( \frac{e^z}{z+i} \right)_{z=i} + 2\pi i \left( \frac{e^z}{z-i} \right)_{z=-i} = 2\pi i \sin(1)$$

\[\square\]

A5.2 Do the following:

(a) Let $\log z$ be the principal branch of the logarithmic function $\log z$. Show that

$$\lim_{x \to x_0 \atop y \to 0^+} \log z - \lim_{x \to x_0 \atop y \to 0^-} \log z = 2\pi i$$

for $z = x + yi \in \mathbb{C} \setminus (-\infty, 0]$ and $x_0 < 0$.

(b) For a piecewise smooth closed curve $\gamma : [0, 1] \to \mathbb{C}^*$, we call the integral

$$\frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z}$$

the winding number of $\gamma$. Show that the winding numbers are always integers. Hint: You may assume that every piecewise smooth curve can be approximated by polygons: for every piecewise smooth curve $\gamma : [0, 1] \to D$ in an open set $D$, there exist a sequence of polygons $\gamma_n : [0, 1] \to D$ such that $\gamma(0) = \gamma_n(0), \gamma(1) = \gamma_n(1)$ and

$$|\gamma(t) - \gamma_n(t)| + |\gamma'(t) - \gamma_n'(t)| < \frac{1}{n}$$

for all $t \in [0, 1]$ and all $n \in \mathbb{Z}^+$. Then you just have to prove the winding numbers of closed polygons are integers.
Proof. For $z = x + yi$ with $y > 0$,

$$\Log z = \ln |z| + i \arccos \frac{x}{|z|}$$

and hence

$$\lim_{x \to x_0 \atop y \to 0^+} \Log z = \lim_{x \to x_0 \atop y \to 0^+} \ln |z| + i \lim_{x \to x_0 \atop y \to 0^+} \arccos \frac{x}{|z|} = \ln |x_0| + \pi i$$

For $z = x + yi$ with $y < 0$,

$$\Log z = \ln |z| - i \arccos \frac{x}{|z|}$$

and hence

$$\lim_{x \to x_0 \atop y \to 0^-} \Log z = \lim_{x \to x_0 \atop y \to 0^-} \ln |z| - i \lim_{x \to x_0 \atop y \to 0^-} \arccos \frac{x}{|z|} = \ln |x_0| - \pi i$$

Consequently,

$$\lim_{x \to x_0 \atop y \to 0^+} \Log z - \lim_{x \to x_0 \atop y \to 0^-} \Log z = 2\pi i.$$  

Let us first prove that the winding numbers are integers for all closed polygons in $\mathbb{C}^*$. Let $\gamma : [0, 1] \to \mathbb{C}^*$ be a closed polygon. Obviously, there exists $c \in \mathbb{C}^*$ such that the line

$$\{tc : t \in \mathbb{R}\}$$

does not pass through any vertex of $\gamma$. Let $\varphi(z) = c^{-1}z$. Then

$$\int_{\varphi \circ \gamma} \frac{dz}{z} = \int_0^1 \frac{d(\varphi(\gamma(t)))}{\varphi(\gamma(t))} = \int_0^1 \frac{\gamma'(t)}{\varphi(\gamma(t))} dt = \int_{\gamma} \frac{dz}{z}$$

So $\varphi \circ \gamma$ and $\gamma$ have the same winding number. And the line

$$\varphi(\{tc : t \in \mathbb{R}\}) = \{c^{-1}tc : t \in \mathbb{R}\} = \{t : t \in \mathbb{R}\}$$

i.e., the real axis does not pass through any vertex of $\varphi \circ \gamma$. So we may replace $\gamma$ by $\varphi \circ \gamma$ and assume that the real axis does not pass through any vertex of $\gamma$.

For simplicity, we may choose $\gamma(0) = \gamma(1)$ not lying on the real axis. Since $\gamma$ is a polygon, it meets the ray $(-\infty, 0]$ at finitely many points. Let us assume that

$$\{t : \text{Im}(\gamma(t)) = 0, \text{Re}(\gamma(t)) < 0\} = \{t_1, t_2, ..., t_m\}$$

for $0 < t_1 < t_2 < ... < t_m < 0$. 

For $\varepsilon > 0$ sufficiently small,

$$\int_{\gamma} \frac{dz}{z} = \int_{0}^{1} \frac{\gamma'(t)}{\gamma(t)} dt = \left( \int_{0}^{t_{1}-\varepsilon} + \int_{t_{1}+\varepsilon}^{t_{2}-\varepsilon} + \ldots + \int_{t_{m}+\varepsilon}^{1} \right) \frac{\gamma'(t)}{\gamma(t)} dt$$

$$= \left( \int_{0}^{t_{1}-\varepsilon} + \int_{t_{m}+\varepsilon}^{1} \right) \frac{\gamma'(t)}{\gamma(t)} dt + \sum_{k=1}^{m-1} \int_{t_{k}+\varepsilon}^{t_{k+1}-\varepsilon} \frac{\gamma'(t)}{\gamma(t)} dt$$

$$+ \sum_{k=1}^{m} \int_{t_{k}-\varepsilon}^{t_{k}+\varepsilon} \frac{\gamma'(t)}{\gamma(t)} dt$$

$$= \log \gamma(t_{1} - \varepsilon) - \log \gamma(0) + \log \gamma(1) - \log \gamma(t_{m} + \varepsilon)$$

$$+ \sum_{k=1}^{m-1} \left( \log \gamma(t_{k+1} - \varepsilon) - \log \gamma(t_{k} + \varepsilon) \right) + \sum_{k=1}^{m} \int_{t_{k}+\varepsilon}^{t_{k}+\varepsilon} \gamma'(t) \gamma(t) dt$$

$$= \sum_{k=1}^{m} \left( \log \gamma(t_{k} - \varepsilon) - \log \gamma(t_{k} + \varepsilon) \right) + \sum_{k=1}^{m} \int_{t_{k}-\varepsilon}^{t_{k}+\varepsilon} \gamma'(t) \gamma(t) dt$$

Since the real axis does not pass through any vertex of $\gamma$,

$$\text{Im}(\gamma(t_{k} - \varepsilon)) \text{Im}(\gamma(t_{k} + \varepsilon)) < 0$$

for each $k$ and $\varepsilon$ sufficiently small. Therefore, by part (a),

$$\lim_{\varepsilon \to 0} \left( \log \gamma(t_{k} - \varepsilon) - \log \gamma(t_{k} + \varepsilon) \right) = \pm 2\pi i$$

for each $k$. For each $k$,

$$\left| \int_{t_{k}-\varepsilon}^{t_{k}+\varepsilon} \frac{\gamma'(t)}{\gamma(t)} dt \right| \leq \frac{1}{M} \int_{t_{k}-\varepsilon}^{t_{k}+\varepsilon} |\gamma'(t)| dt = \frac{2\varepsilon |\gamma'(t_{k})|}{M}$$

where $M = \min |\gamma(t)|$ for $t \in [0, 1]$. Therefore,

$$\lim_{\varepsilon \to 0} \int_{t_{k}-\varepsilon}^{t_{k}+\varepsilon} \frac{\gamma'(t)}{\gamma(t)} dt = 0$$

for $k = 1, 2, \ldots, m$. In conclusion,

$$\int_{\gamma} \frac{dz}{z} = \sum_{k=1}^{m} \lim_{\varepsilon \to 0} \left( \log \gamma(t_{k} - \varepsilon) - \log \gamma(t_{k} + \varepsilon) \right)$$

$$+ \sum_{k=1}^{m} \lim_{\varepsilon \to 0} \int_{t_{k}-\varepsilon}^{t_{k}+\varepsilon} \frac{\gamma'(t)}{\gamma(t)} dt = 2N\pi i$$

for some integer $N$. So the winding numbers of all closed polygons $\gamma : [0, 1] \to \mathbb{C}^*$ are integers.
Finally, let us prove the statement for an arbitrary piecewise smooth closed curve $\gamma : [0,1] \to \mathbb{C}$. We can approximate $\gamma$ by a sequence of closed polygons $\gamma_n$ as in the hint. Then

$$\left| \int_\gamma \frac{dz}{z} - \int_{\gamma_n} \frac{dz}{z} \right| = \left| \int_0^1 \frac{\gamma'(t)}{\gamma(t)} \, dt - \int_0^1 \frac{\gamma_n'(t)}{\gamma_n(t)} \, dt \right|$$

$$= \left| \int_0^1 \left( \frac{\gamma'(t)}{\gamma(t)} - \frac{\gamma_n'(t)}{\gamma_n(t)} \right) \, dt + \int_0^1 \left( \frac{\gamma_n'(t)}{\gamma_n(t)} - \frac{\gamma_n'(t)}{\gamma_n(t)} \right) \, dt \right|$$

$$\leq \int_0^1 \left| \frac{\gamma'(t)}{\gamma(t)} - \frac{\gamma_n'(t)}{\gamma_n(t)} \right| \, dt + \int_0^1 \left| \frac{\gamma_n'(t)}{\gamma_n(t)} - \frac{\gamma_n'(t)}{\gamma_n(t)} \right| \, dt$$

We let $M = \min |\gamma(t)|$ for $t \in [0,1]$. Since $|\gamma(t) - \gamma_n(t)| < 1/n$, we have

$$|\gamma_n(t)| > M - \frac{1}{n}, \quad \frac{1}{|\gamma_n(t)|} < \frac{n}{nM - 1}$$

and

$$\left| \frac{1}{\gamma(t)} - \frac{1}{\gamma_n(t)} \right| = \frac{|\gamma_n(t) - \gamma(t)|}{|\gamma(t)\gamma_n(t)|} < \frac{1}{M(nM - 1)}$$

for all $t \in [0,1]$. Therefore,

$$\left| \int_\gamma \frac{dz}{z} - \int_{\gamma_n} \frac{dz}{z} \right| \leq \int_0^1 \left| \frac{\gamma'(t)}{\gamma(t)} - \frac{\gamma_n'(t)}{\gamma_n(t)} \right| \, dt + \int_0^1 \left| \frac{\gamma_n'(t)}{\gamma_n(t)} - \frac{\gamma_n'(t)}{\gamma_n(t)} \right| \, dt$$

$$< \frac{1}{M(nM - 1)} \int_0^1 |\gamma'(t)| \, dt + \frac{n}{nM - 1} \int_0^1 |\gamma'(t) - \gamma_n'(t)| \, dt$$

$$< \frac{1}{M(nM - 1)} \int_0^1 |\gamma'(t)| \, dt + \frac{n}{nM - 1} \int_0^1 \frac{1}{n} \, dt$$

$$= \frac{1}{M(nM - 1)} \int_0^1 |\gamma'(t)| \, dt + \frac{1}{nM - 1}$$

which goes to 0 as $n \to \infty$. Consequently,

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z} = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\gamma_n} \frac{dz}{z}$$

must be an integer since

$$\frac{1}{2\pi i} \int_{\gamma_n} \frac{dz}{z}$$

are integers for all $n$. \hfill \square

A5.3 Let $f(z)$ be an analytic function on a simply connected open set $D$. Show that if $f(z)$ does not vanish on $D$, then

(a) there exists an analytic function $g(z)$ such that $e^{g(z)} = f(z)$ on $D$;
(b) for every nonzero integer \( n \), there exists an analytic function \( h(z) \) such that \( (h(z))^n = f(z) \) on \( D \).

Hint: consider the anti-derivatives of \( f'(z)/f(z) \).

**Proof.** Since \( D \) is simply connected, every analytic function on \( D \) has an anti-derivative.

We let \( g(z) \) be the anti-derivative of \( f'(z)/f(z) \) such that \( e^{g(z_0)} = f(z_0) \) for some \( z_0 \in D \). Let \( \varphi(z) = \exp(g(z)) \). Then

\[
\frac{\varphi'(z)}{\varphi(z)} = \frac{f'(z)}{f(z)} \Rightarrow \frac{\varphi'(z)f(z) - \varphi(z)f'(z)}{\varphi(z)f(z)} = 0 \Rightarrow \left( \frac{\varphi(z)}{f(z)} \right)' = 0
\]

on \( D \). Therefore, \( \varphi(z)/f(z) = c \) is constant on \( D \). And since \( \varphi(z_0) = f(z_0) \), \( c = 1 \). Therefore, \( f(z) = \varphi(z) = \exp(g(z)) \) on \( D \).

We let \( h(z) = \exp(g(z)/n) \). Then \( (h(z))^n = \exp(g(z)) = f(z) \) on \( D \).

\[\square\]

A5.4 Show that

\[
\lim_{n \to \infty} \sum_{m=-n}^{n} \int_{|z-m\pi|=1} \frac{dz}{z^2 \sin z} = 0
\]

where all circles are oriented in the same direction. Hint: consider the integral

\[
\int_{\gamma_n} \frac{dz}{z^2 \sin z}
\]

along the boundary \( \gamma_n \) of the square \( \{|x|, |y| \leq n\pi + \frac{\pi}{2}\} \) for \( n = 1, 2, ..., \).

**Proof.** Since \((z^2 \sin z)^{-1}\) is analytic in

\[
\left\{ x + yi : |x|, |y| \leq n\pi + \frac{\pi}{2} \right\} \setminus \{m\pi : m \in \mathbb{Z}, |m| \leq n\}
\]

we have

\[
\int_{\gamma_n} \frac{dz}{z^2 \sin z} = \sum_{m=-n}^{n} \int_{|z-m\pi|=1} \frac{dz}{z^2 \sin z}
\]

by CIT. It suffices to show that

\[
\lim_{n \to \infty} \int_{\gamma_n} \frac{dz}{z^2 \sin z} = 0.
\]

Since

\[
|\sin z|^2 = (\sin x)^2 + \left( \frac{e^y - e^{-y}}{2} \right)^2
\]
for \( z = x + yi \), we have
\[
|\sin z|^2 \geq \left( \sin \left( n\pi + \frac{\pi}{2} \right) \right)^2 = 1 \text{ if } |x| = n\pi + \frac{\pi}{2}
\]
\[
|\sin z|^2 \geq \left( \frac{e^{n\pi + \pi/2} - e^{-n\pi - \pi/2}}{2} \right)^2 > 1 \text{ if } |y| = n\pi + \frac{\pi}{2}
\]
Therefore, \( |\sin z| \geq 1 \) for all \( z \in \gamma_n \). Consequently,
\[
\left| \int_{\gamma_n} \frac{dz}{z^2 \sin z} \right| \leq (8n + 4)\pi \max_{z \in \gamma_n} \frac{1}{|z|^2 |\sin z|}
\]
\[
\leq (8n + 4)\pi \left( \frac{4}{(2n + 1)^2 \pi^2} \right) = \frac{16}{(2n + 1)\pi}
\]
which goes to 0 as \( n \to \infty \). Therefore,
\[
\lim_{n \to \infty} \sum_{m=-n}^{n} \int_{|z-m\pi|=1} \frac{dz}{z^2 \sin z} = \lim_{n \to \infty} \int_{\gamma_n} \frac{dz}{z^2 \sin z}.
\]

A5.5 Let \( D \) be the annulus \( \{1 < |z| < 3\} \).
(a) Show that an analytic function \( f(z) \) has an anti-derivative on \( D \) if and only if
\[
\int_{|z|=2} f(z) \, dz = 0.
\]
(b) Let \( V \) be the complex vector space of all analytic functions on \( D \) and let \( T : V \to V \) be the linear transformation given by \( T(f(z)) = f'(z) \). Show that
\[
\frac{V}{T(V)} \cong \mathbb{C}.
\]
Proof. If \( f(z) \) has an anti-derivative in \( D \), \( \int_{\gamma} f(z) \, dz = 0 \) for all piecewise smooth closed curve \( \gamma \) in \( D \). Hence \( \int_{|z|=2} f(z) \, dz = 0 \).
Suppose that \( \int_{|z|=2} f(z) \, dz = 0 \). Let \( D_1 = D \cap \{\text{Im}(z) > -1/2\} \) and \( D_2 = D \cap \{\text{Im}(z) < 1/2\} \).
Both \( D_1 \) and \( D_2 \) are simply connected. So \( f(z) \) has anti-derivatives \( F_1(z) \) and \( F_2(z) \) in \( D_1 \) and \( D_2 \), respectively. We
may choose $F_j(z)$ such that $F_1(-2) = F_2(-2)$. The intersection $D_1 \cap D_2$ has two connected components $G_1$ and $G_2$, where

$G_1 = D \cap \{-1/2 < \text{Im}(z) < 1/2, \text{Re}(z) > 0\}$ and

$G_2 = D \cap \{-1/2 < \text{Im}(z) < 1/2, \text{Re}(z) < 0\}$.

On each $G_j$, $(F_1(z) - F_2(z))' = 0$ and hence $F_1(z) - F_2(z)$ is constant. Since $F_1(-2) = F_2(-2)$, $F_1(z) = F_2(z)$ on $G_2$.

Since

$$\int_{|z|=2} f(z) \, dz = \int_{|z|=2, \text{Im}(z)\geq 0} f(z) \, dz + \int_{|z|=2, \text{Im}(z)\leq 0} f(z) \, dz$$

$$= F_1(-2) - F_1(2) + F_2(2) - F_2(-2)$$

$$= F_2(2) - F_1(2)$$

we conclude that $F_2(2) = F_1(2)$. Therefore, $F_1(z) = F_2(z)$ on $G_1$.

In conclusion, $F_1(z) = F_2(z)$ for all $z \in D_1 \cap D_2$. Then

$$F(z) = \begin{cases} 
    F_1(z) & \text{if } z \in D_1 \\
    F_2(z) & \text{if } z \in D_2
\end{cases}$$

is an anti-derivative of $f(z)$ on $D$.

We let $\rho : V \to \mathbb{C}$ be the map

$$\rho(f(z)) = \int_{|z|=2} f(z) \, dz$$

We claim that $\rho$ is a surjective linear transformation. It is a linear transformation because

$$\rho(f(z) + cg(z)) = \int_{|z|=2} (f(z) + cg(z)) \, dz$$

$$= \int_{|z|=2} f(z) \, dz + c \int_{|z|=2} g(z) \, dz$$

$$= \rho(f(z)) + c\rho(g(z))$$

for all $f(z), g(z) \in V$ and $c \in \mathbb{C}$. It is surjective since

$$\rho(cz^{-1}) = \int_{|z|=2} \frac{c}{z} \, dz = 2\pi i c$$

for all $c \in \mathbb{C}$.

By part (a), the kernel $\ker(\rho)$ consists of analytic functions $f(z) \in V$ such that $f(z) = F'(z)$ for some $F(z) \in V$. In other words, $\ker(\rho) = T(V)$. 
In conclusion, $\rho : V \rightarrow \mathbb{C}$ is a surjective linear transformation with $\ker(\rho) = T(V)$ and hence

$$\frac{V}{T(V)} \cong \frac{V}{\ker(\rho)} \cong \mathbb{C}.$$