A2.1 For each of the following functions $f : \mathbb{C} \rightarrow \mathbb{C}$, find where $f(z)$ is complex differentiable and where $f(z)$ is analytic. You must justify your answer.

(a) $f(z) = e^{x^2 - y^2} (\cos(2xy) + i \sin(2xy))$

(b) $f(z) = z^2 + \bar{z}^2$

(c) $f(z) = \begin{cases} 
\frac{z^2 - 1}{z - 1} & \text{if } z \neq 1 \\
2 & \text{if } z = 1 
\end{cases}$

(d) $f(z) = \begin{cases} 
\frac{z^3}{|z|} & \text{if } z \neq 0 \\
0 & \text{if } z = 0 
\end{cases}$

Here $z = x + yi$.

**Solution.**

(a) Clearly, $e^{x^2 - y^2} \cos(2xy)$ and $e^{x^2 - y^2} \sin(2xy)$ have continuous partial derivatives everywhere on $\mathbb{R}^2$ and are hence totally differentiable everywhere. Since

$$
\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (e^{x^2 - y^2} (\cos(2xy) + i \sin(2xy)))
$$

$$
= (\cos(2xy) + i \sin(2xy)) \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) e^{x^2 - y^2}
$$

$$
+ e^{x^2 - y^2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (\cos(2xy) + i \sin(2xy))
$$

$$
= e^{x^2 - y^2} (2x - 2yi)(\cos(2xy) + i \sin(2xy))
$$

$$
+ e^{x^2 - y^2} (-2y \sin(2xy) - 2xi \sin(2xy) + 2yi \cos(2xy) - 2x \cos(2xy)) = 0
$$

everywhere on $\mathbb{C}$, $f$ is complex differentiable and analytic everywhere.

(b) Clearly,

$$f(z) = z^2 + \bar{z}^2 = 2(x^2 - y^2)$$

has continuous partial derivatives everywhere and are hence totally differentiable on $\mathbb{C}$. Since

$$
\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f = 4x - 4yi
$$
only vanishes at \((0,0)\), \(f(z)\) is only complex differentiable at \((0,0)\). Since \(f(z)\) is not complex differentiable in any disk centered at \((0,0)\), \(f(z)\) is not analytic at \((0,0)\). Therefore, \(f(z)\) is nowhere analytic.

(c) Clearly, \(f(z) = z + 1\) for all \(z \in \mathbb{C}\). Since \(z + 1\) is a polynomial in \(z\), it is complex differentiable and analytic everywhere.

(d) For \(z \neq 0\),

\[
\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f = \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \frac{x^3}{\sqrt{x^2 + y^2}} = \frac{2x^4 + 3x^2y^2}{(x^2 + y^2)^{3/2}} - i \frac{x^3y}{(x^2 + y^2)^{3/2}}
\]

is continuous and only vanish at \(x = 0\). So \(f(z)\) is complex differentiable at \(\{ z : \text{Re}(z) = 0, z \neq 0 \}\).

At \(z = 0\), since

\[
0 \leq \frac{|x|^3}{x^2 + y^2} \leq |x|
\]

we have

\[
\lim_{z \to 0} \frac{|f(x,y) - f(0,0) - 0(x - 0) - 0(y - 0)|}{||(x, y) - (0,0)||} = \lim_{z \to 0} \frac{|x|^3}{x^2 + y^2} = 0
\]

So \(f\) is totally differentiable at \(z = 0\) with

\[
\left. \frac{\partial f}{\partial x} \right|_0 = \left. \frac{\partial f}{\partial y} \right|_0 = 0
\]

\[
\Rightarrow \left. \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right) \right|_0 = 0.
\]

Therefore, \(f(z)\) is complex differentiable at 0.

In conclusion, \(f(z)\) is only complex differentiable on the imaginary axis \(\{ z : \text{Re}(z) = 0 \}\). But for every \(z\) with \(\text{Re}(z) = 0\), \(f(z)\) is not complex differentiable in any disk at \(z\). So \(f(z)\) is nowhere analytic. \(\square\)

A2.2 Show that under the change of coordinates \(x = r \cos \theta\) and \(y = r \sin \theta\),

\[
\frac{\partial}{\partial r} = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial \theta} = -r \sin \theta \frac{\partial}{\partial x} + r \cos \theta \frac{\partial}{\partial y}
\]

Derive the Cauchy-Riemann equations under polar coordinates.
Solution. For every differentiable function \( f(x, y) \),

\[
\frac{\partial f}{\partial r} = \frac{\partial (r \cos \theta)}{\partial r} \frac{\partial f}{\partial x} + \frac{\partial (r \sin \theta)}{\partial r} \frac{\partial f}{\partial y} = \cos \theta \frac{\partial f}{\partial x} + \sin \theta \frac{\partial f}{\partial y},
\]

\[
\frac{\partial f}{\partial \theta} = \frac{\partial (r \cos \theta)}{\partial \theta} \frac{\partial f}{\partial x} + \frac{\partial (r \sin \theta)}{\partial \theta} \frac{\partial f}{\partial y} = -r \sin \theta \frac{\partial f}{\partial x} + r \cos \theta \frac{\partial f}{\partial y}
\]

by chain rule. Therefore,

\[
\frac{\partial}{\partial r} = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial \theta} = -r \sin \theta \frac{\partial}{\partial x} + r \cos \theta \frac{\partial}{\partial y}
\]

It follows that

\[
\begin{bmatrix}
\frac{\partial}{\partial r} \\
\frac{\partial}{\partial \theta}
\end{bmatrix}
= \begin{bmatrix}
\cos \theta & \sin \theta \\
-r \sin \theta & r \cos \theta
\end{bmatrix}
\begin{bmatrix}
\frac{\partial f}{\partial x} \\
\frac{\partial f}{\partial y}
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial f}{\partial x} \\
\frac{\partial f}{\partial y}
\end{bmatrix}
= \frac{1}{r} \begin{bmatrix}
r \cos \theta & -r \sin \theta \\
r \sin \theta & r \cos \theta
\end{bmatrix}
\begin{bmatrix}
\frac{\partial}{\partial r} \\
\frac{\partial}{\partial \theta}
\end{bmatrix}
\]

Therefore,

\[
\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} = \frac{1}{r} \left( r \cos \theta \frac{\partial}{\partial r} - \sin \theta \frac{\partial}{\partial \theta} \right) + i \frac{1}{r} \left( r \sin \theta \frac{\partial}{\partial r} + \cos \theta \frac{\partial}{\partial \theta} \right)
\]

So we obtain the CR equations:

\[
\begin{cases}
r \cos \theta \frac{\partial u}{\partial r} - \sin \theta \frac{\partial u}{\partial \theta} = r \sin \theta \frac{\partial v}{\partial r} + \cos \theta \frac{\partial v}{\partial \theta} \\
r \sin \theta \frac{\partial u}{\partial r} + \cos \theta \frac{\partial u}{\partial \theta} = -r \cos \theta \frac{\partial v}{\partial r} + \sin \theta \frac{\partial v}{\partial \theta}
\end{cases}
\]

or equivalently,

\[
\begin{cases}
\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} \\
\frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r}
\end{cases}
\]

\[\square\]

A2.3 Let \( f : D \to \mathbb{C} \) be a complex function on a domain (connected and open set) \( D \subset \mathbb{C} \). Show that if both \( f(z) \) and \( \overline{f(z)} \) are analytic on \( D \), then \( f(z) \) is constant on \( D \).
Proof. Let \( f(z) = u(x, y) + iv(x, y) \) for \( u = \operatorname{Re}(f) \) and \( v = \operatorname{Im}(f) \). Since both \( f(z) \) and \( \overline{f(z)} \) are analytic on \( D \), we have
\[
\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f = 0 \Rightarrow \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (u + iv) = 0
\]
\[
\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \overline{f} = 0 \Rightarrow \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (u - iv) = 0
\]
\[
\Rightarrow \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) u = \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) v = 0
\]
\[
\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0 \Rightarrow f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = 0
\]
on \( D \). And since \( D \) is connected, \( f(z) \) is constant on \( D \). \( \square \)

A2.4 For a differentiable map \( f : U \to \mathbb{R}^m \) on an open set \( U \subset \mathbb{R}^n \), the Jacobian of \( f \) is the \( m \times n \) matrix
\[
\left[ \frac{\partial f_i}{\partial x_j} \right]_{m \times n}
\]
if \( f \) is given by
\[
f(x_1, x_2, ..., x_n) = (f_1(x_1, x_2, ..., x_n), f_2(x_1, x_2, ..., x_n), ..., f_m(x_1, x_2, ..., x_n)).
\]

Let \( f : D \to \mathbb{C} \) be a complex function on an open set \( D \subset \mathbb{C} \). Show that if \( f(z) \) is complex differentiable at \( z_0 \in D \), then
\[
\det J_f(z_0) = |f'(z_0)|^2
\]

Proof. Let \( f(z) = u(x, y) + iv(x, y) \) for \( u = \operatorname{Re}(f) \) and \( v = \operatorname{Im}(f) \). Then
\[
J_f = \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{bmatrix} \Rightarrow \det(J_f) = \left( \frac{\partial u}{\partial x} \right) \left( \frac{\partial v}{\partial y} \right) - \left( \frac{\partial u}{\partial y} \right) \left( \frac{\partial v}{\partial x} \right)
\]
When \( f(z) \) is complex differentiable at \( z_0 \),
\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{and} \quad f'(z_0) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}
\]
at \( z_0 \). Therefore,
\[
\det J_f(z_0) = \left( \frac{\partial u}{\partial x} \right) \left( \frac{\partial v}{\partial y} \right) - \left( \frac{\partial u}{\partial y} \right) \left( \frac{\partial v}{\partial x} \right) \\
= \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \\
= |f'(z_0)|^2.
\]
□

A2.5 For a sequence \( \{a_n : n = 0, 1, 2, \ldots \} \) of complex numbers, show that
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \geq \lim_{n \to \infty} \sqrt[\infty]{|a_n|}
\]
and the equality holds if the limit
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|
\]
exists. You may assume that \( a_n \neq 0 \) for all \( n \).

Proof. We let \( b_n = \ln |a_n| - \ln |a_{n-1}| \) for \( n \geq 1 \) and \( b_0 = \ln |a_0| \).
Then
\[
\ln \sqrt[n]{|a_n|} = \frac{1}{n} \ln |a_n| = \frac{1}{n} \sum_{m=0}^{n} b_n
\]
\[
\lim_{n \to \infty} \sqrt[n]{|a_n|} = \exp \left( \lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n} b_n \right)
\]
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \exp \left( \lim_{n \to \infty} b_n \right)
\]
So to show
\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \geq \lim_{n \to \infty} \sqrt[n]{|a_n|}
\]
it suffices to show that
\[
\lim_{n \to \infty} b_n \geq \lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n} b_n
\]
Suppose that
\[
L = \lim_{n \to \infty} b_n.
\]
Then for every $u > L$, there exists $N$ such that $b_n \leq u$ for all $n > N$. Fixing $u$ and $N$, we have

\[
\frac{1}{n} \sum_{m=0}^{n} b_n = \frac{1}{n} \sum_{m=0}^{N} b_m + \frac{1}{n} \sum_{m=N+1}^{n} b_m
\]

\[
\leq \frac{1}{n} \sum_{m=0}^{N} b_m + \left( \frac{n - N}{n} \right) u
\]

\[
\Rightarrow \lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n} b_n \leq \lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{N} b_m + \lim_{n \to \infty} \left( \frac{n - N}{n} \right) u = u
\]

That is,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n} b_n \leq u
\]

for all $u > L$. Therefore,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n} b_n \leq L = \lim_{n \to \infty} b_n.
\]

Replacing $b_n$ by $-b_n$, we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n} (-b_n) \leq \lim_{n \to \infty} (-b_n)
\]

\[
\Rightarrow -\lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n} b_n \leq - \lim_{n \to \infty} b_n
\]

\[
\Rightarrow \lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n} b_n \geq \lim_{n \to \infty} b_n
\]

In conclusion, we have

\[
\lim_{n \to \infty} b_n \geq \lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n} b_n \geq \lim_{n \to \infty} \frac{1}{n} \sum_{m=0}^{n} b_n \geq \lim_{n \to \infty} b_n
\]

Taking exp, we obtain

\[
\lim_{n \to \infty} \sqrt[n]{a_{n+1}} \geq \lim_{n \to \infty} \sqrt[n]{a_n} \geq \lim_{n \to \infty} \sqrt[n]{a_n} \geq \lim_{n \to \infty} \frac{a_{n+1}}{a_n}.
\]

When

\[
\lim_{n \to \infty} \sqrt[n]{a_{n+1}} = \lim_{n \to \infty} \sqrt[n]{a_n} = \lim_{n \to \infty} \sqrt[n]{a_n} = \lim_{n \to \infty} \frac{a_{n+1}}{a_n}.
\]
we have

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|. $$