Solutions for Math 411 Assignment #1

A1.1 For each of the following subsets $D$ of $\mathbb{R}^2$, is $D$ open, closed and/or compact? You must justify your answer.

(a) $D = \{(x, y) : |x| + |y| \leq 1\}$.
(b) $D = \{(x, y) : \sin x + \cos y > 1\}$.
(c) $D = \{(n, \frac{1}{n}) : n \in \mathbb{Z}, n > 0\}$.
(d) $D = \{(x, y) : x^4 + y^2 = 1\}$.

Solution. (a) Let $f(x, y) = |x| + |y|$. Since $|x|$ and $|y|$ are continuous functions on $\mathbb{R}$, $f(x, y)$ is continuous on $\mathbb{R}^2$. So $D = f^{-1}(I)$ is closed for the closed set $I = (-\infty, 1]$. For $|x| + |y| \leq 1$,

$$x^2 + y^2 \leq (|x| + |y|)^2 \leq 1.$$ 

So $D \subset \{x^2 + y^2 \leq 1\}$ is bounded. And since $(0, 0) \in D$, $D \neq \emptyset$. Since $D \neq \emptyset$, $D \neq \mathbb{R}^2$ and $\mathbb{R}^2$ is connected, $D$ cannot be both open and closed. So $D$ is not open. Since $D$ is closed and bounded, $D$ is compact.

(b) Let $f(x, y) = \sin x + \cos y$. Since $\sin x$ and $\cos y$ are continuous on $\mathbb{R}$, $f(x, y)$ is continuous on $\mathbb{R}^2$. So $D = f^{-1}(I)$ is open for the open set $I = (1, \infty)$. Since $(\pi/2, 0) \in D$ and $(0, \pi/2) \notin D$, $D \neq \emptyset$ and $D \neq \mathbb{R}^2$. Since $D \neq \emptyset$, $D \neq \mathbb{R}^2$ and $\mathbb{R}^2$ is connected, $D$ cannot be both open and closed. So $D$ is not closed. Hence $D$ is not compact.

(c) We claim that $D$ is closed. Otherwise, there exists a point $p = (x_0, y_0) \notin D$ and a sequence of points $\{(n_j, n_j^{-1})\} \subset D$ such that

$$\lim_{j \to \infty} n_j = x_0 \text{ and } \lim_{j \to \infty} n_j^{-1} = y_0.$$ 

Since $\lim n_j$ exists and $n_j$ are integers, we must have

$$n_k = n_{k+1} = \ldots = x_0$$

for some $k$. So $y_0 = \lim n_j^{-1} = 1/n_k$ and $x_0 = n_k$. Then $(x_0, y_0) \in D$. Contradiction. So $D$ is closed. Since $(1, 1) \in D$ and $(0, 0) \notin D$, $D \neq \emptyset$ and $D \neq \mathbb{R}^2$. Since $D \neq \emptyset$, $D \neq \mathbb{R}^2$ and $\mathbb{R}^2$ is connected, $D$ cannot be both open and closed. So $D$ is not open. Since $\lim_{n \to \infty} ||(n, 1/n)|| = \infty$, $D$ is unbounded. So $D$ is not compact.

(d) Let $f(x, y) = x^4 + y^2$. Since $x^4$ and $y^2$ are continuous functions on $\mathbb{R}$, $f(x, y)$ is continuous on $\mathbb{R}^2$. So $D = f^{-1}\{1\}$ is closed. Since $(1, 0) \in D$ and $(0, 0) \notin D$, $D \neq \emptyset$ and $D \neq \mathbb{R}^2$.

\[http://www.math.ualberta.ca/~xichen/math41117f/hw1sol.pdf\]
Since $D \neq \emptyset$, $D \neq \mathbb{R}^2$ and $\mathbb{R}^2$ is connected, $D$ cannot be both open and closed. So $D$ is not open. For $(x, y) \in D$,

$$x^4 + y^2 = 1 \Rightarrow |x| \leq 1, |y| \leq 1 \Rightarrow x^2 + y^2 \leq 2.$$ 

So $D \subset \{x^2 + y^2 \leq 2\}$ is bounded. Since $D$ is closed and bounded, $D$ is compact. □

A1.2 Let $f(z) \in \mathbb{R}[z]$ be a polynomial in $z$ with real coefficients. Show that if $z_0$ is a complex root of $f(z)$, so is $\overline{z}_0$.

Proof. Let $f(z) = a_0 + a_1 z + \ldots + a_n z^n$ for $a_j \in \mathbb{R}$. Then

$$f(z_0) = 0 \Rightarrow \overline{f(z_0)} = 0$$

$$\Rightarrow \sum_{j=0}^{n} a_j z_0^j = 0$$

$$\Rightarrow \sum_{j=0}^{n} \overline{a_j} \overline{z}_0^j = 0$$

$$\Rightarrow \sum_{j=0}^{n} a_j \overline{z}_0^j = 0$$

$$\Rightarrow f(\overline{z}_0) = 0$$

since $a_j$ are real. So $\overline{z}_0$ is a root of $f(z)$. □

A1.3 Show that for all $z_1, z_2, z_3 \in \mathbb{C}$,

$$2|z_1| + 2|z_2| + 2|z_3| \geq |z_1 + z_2| + |z_2 + z_3| + |z_3 + z_1|.$$ 

Also find the necessary and sufficient conditions for the equality to hold.

Proof. By triangle inequality,

$$|z_1| + |z_2| \geq |z_1 + z_2|$$

$$|z_2| + |z_3| \geq |z_2 + z_3|$$

$$|z_3| + |z_1| \geq |z_3 + z_1|$$

Summing them up, we obtain

$$2|z_1| + 2|z_2| + 2|z_3| \geq |z_1 + z_2| + |z_2 + z_3| + |z_3 + z_1|.$$ 

The equality holds if and only if

$$|z_1| + |z_2| = |z_1 + z_2|$$

$$|z_2| + |z_3| = |z_2 + z_3|$$

$$|z_3| + |z_1| = |z_3 + z_1|$$
which happens when $z_j$ and $z_k$ are in the same direction as vectors for all pairs $j$ and $k$. Therefore,

$$2|z_1| + 2|z_2| + 2|z_3| = |z_1 + z_2| + |z_2 + z_3| + |z_3 + z_1|.$$ 

if and only if there exist $z_0 \in \mathbb{C}$ and $a_1, a_2, a_3 \geq 0$ such that $z_j = a_j z_0$ for $j = 1, 2, 3$. \hfill \Box

A1.4 Show that every linear transformation $T : \mathbb{C} \rightarrow \mathbb{C}$, as vector spaces over $\mathbb{R}$, is given by

$$T(z) = az + b\overline{z}$$

for some complex numbers $a$ and $b$.

Proof. Every linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is given by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

for some $2 \times 2$ real matrix $[c_{ij}]$. Therefore,

$$T(z) = T(x + yi) = (c_{11}x + c_{12}y) + i(c_{21}x + c_{22}y)$$

$$= (c_{11} + c_{21}i)x + (c_{12} + c_{22}i)y$$

$$= (c_{11} + c_{21}i)\frac{z + \overline{z}}{2} + (c_{12} + c_{22}i)\frac{z - \overline{z}}{2i}$$

$$= \left(\frac{c_{11} + c_{21}i}{2} + \frac{c_{12} + c_{22}i}{2i}\right)z + \left(\frac{c_{11} + c_{21}i}{2} - \frac{c_{12} + c_{22}i}{2i}\right)\overline{z}$$

$$= az + b\overline{z}$$ \hfill \Box

A1.5 Let $\Delta ABC$ be a triangle in the complex plane whose vertices $A$, $B$, $C$ are given by $a, b, c \in \mathbb{C}$. Show that the area of $\Delta ABC$ is given by

$$S_{\Delta ABC} = \frac{1}{2} \left| \text{Im}((a - b)(a - c)) \right|.$$ 

Proof. Let

$$a - b = r_1 (\cos \theta_1 + i \sin \theta_2)$$

$$a - c = r_2 (\cos \theta_2 + i \sin \theta_2).$$
So the angle $BAC$ is $\pm (\theta_1 - \theta_2)$ and the area of the triangle is

$$S_{\Delta ABC} = \frac{1}{2} |AB||AC| |\sin(\theta_1 - \theta_2)|$$

$$= \frac{1}{2} |a - b||a - c| |\sin(\theta_1 - \theta_2)|$$

$$= \frac{1}{2} |r_1 r_2 \sin(\theta_1 - \theta_2)|$$

$$= \frac{1}{2} |\text{Im}(r_1 r_2 \cos(\theta_1 - \theta_2) + i r_1 r_2 \sin(\theta_1 - \theta_2))|$$

$$= \frac{1}{2} |\text{Im}(r_1 (\cos \theta_1 + i \sin \theta_1) r_2 (\cos \theta_2 - i \sin \theta_2))|$$

$$= \frac{1}{2} |\text{Im}((a - b)(a - c))|.$$