Math 411 Final Solutions

(1) Prove the Fundamental Theorem of Algebra: every complex polynomial in one variable of degree at least one has a complex root.

Proof. Let \( f(z) \) be a complex polynomial of degree at least one. Suppose that \( f(z) \) has no complex roots. Then \( g(z) = 1/f(z) \) is an entire function. Since \( \deg f(z) \geq 1 \),

\[
\lim_{z \to \infty} g(z) = \lim_{z \to \infty} \frac{1}{f(z)} = 0.
\]

Then there exists \( R > 0 \) such that \( |g(z)| \leq 1 \) for \( |z| > R \). Hence

\[
|g(z)| \leq \max(M, 1)
\]

for all \( z \) and

\[
M = \max_{|z| \leq R} |g(z)|.
\]

By Louville’s Theorem, \( g(z) \equiv c \) for some \( c \in \mathbb{C} \). It follows that \( f(z) \equiv 1/c \). Contradiction. \( \square \)

(2) Let \( f(z) \) be a nowhere vanishing analytic function on a simply connected open set \( D \). Prove the following:

(a) For every nonzero integer \( n \), there exists an analytic function \( h(z) \) such that \( (h(z))^n = f(z) \) on \( D \).

(b) If there are two analytic functions \( h_1(z) \) and \( h_2(z) \) on \( D \) such that \( (h_j(z))^n \equiv f(z) \) for \( j = 1, 2 \) and a nonzero integer \( n \), then there exists a constant \( c \in \mathbb{C} \) such that

\[
c^n = 1 \quad \text{and} \quad h_1(z) \equiv ch_2(z)
\]

on \( D \).

Proof. Since \( D \) is simply connected, every analytic function on \( D \) has an anti-derivative.

We let \( g(z) \) be the anti-derivative of \( f'(z)/f(z) \) such that \( e^{g(z_0)} = f(z_0) \) for some \( z_0 \in D \). Let \( \varphi(z) = \exp(g(z)) \). Then

\[
\frac{\varphi'(z)}{\varphi(z)} = \frac{f'(z)}{f(z)} \Rightarrow \frac{\varphi'(z)f(z) - \varphi(z)f'(z)}{\varphi(z)f(z)} = 0 \Rightarrow \left( \frac{\varphi(z)}{f(z)} \right)' = 0
\]

on \( D \). Therefore, \( \varphi(z)/f(z) = c \) is constant on \( D \). And since \( \varphi(z_0) = f(z_0) \), \( c = 1 \). Therefore, \( f(z) = \varphi(z) = \exp(g(z)) \) on \( D \).
We let \( h(z) = \exp(g(z)/n) \). Then \( (h(z))^n = \exp(g(z)) = f(z) \) on \( D \).

If \( (h_1(z))^n \equiv (h_2(z))^n \equiv f(z) \) on \( D \),

\[
\left( \frac{h_1(z)}{h_2(z)} \right)^n \equiv 1
\]
as \( h_j(z) \) does not vanish on \( D \). Therefore,

\[
\left( \frac{h_1(z)}{h_2(z)} \right)^n \equiv 0 \Rightarrow n \left( \frac{h_1(z)}{h_2(z)} \right)^{n-1} \left( \frac{h_1(z)}{h_2(z)} \right) \equiv 0
\]

\[
\Rightarrow \left( \frac{h_1(z)}{h_2(z)} \right) \equiv 0 \Rightarrow \frac{h_1(z)}{h_2(z)} \equiv c
\]

and hence \( h_1(z) \equiv ch_2(z) \) for some constant \( c \) on \( D \). And since \( (h_1(z))^n \equiv (h_2(z))^n, c^n = 1 \). \( \square \)

(3) For each of the following functions, do the following:

- find all its singularities in \( \mathbb{C} \);
- find the principal part of the function at each singularity;
- for each singularity, determine whether it is a pole, a removable singularity, or an essential singularity;
- compute the residue of the function at each singularity.

(a) \( f(z) = (1 - z) \exp \left( \frac{1}{z^2} \right) \)

(b) \( f(z) = \frac{(\tan z)^2}{z} \)

(c) \( f(z) = \frac{e^z}{z(z - 1)^2} \)

\textit{Solution.} (a) \( f(z) \) has a singularity at 0. At \( z = 0 \),

\[
(1 - z) \exp \left( \frac{1}{z^2} \right) = (1 - z)e^{1/z^2} = (1 - z) \sum_{n=0}^{\infty} \frac{1}{(n!)z^{2n}}
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{(n!)z^{2n}} - \sum_{n=0}^{\infty} \frac{1}{(n!)z^{2n-1}}
\]

\[
= 1 - z + \sum_{n=1}^{\infty} \frac{1}{(n!)z^{2n}} - \sum_{n=1}^{\infty} \frac{1}{(n!)z^{2n-1}}
\]
in \( 0 < |z| < \infty \). So the principal part of \( f(z) \) at 0 is

\[
\sum_{n=1}^{\infty} \frac{1}{(n!)z^{2n}} - \sum_{n=1}^{\infty} \frac{1}{(n!)z^{2n-1}}
\]
Consequently, \( f(z) \) has an essential singularity at 0 and
\[
\text{Res}_{z=0} f(z) = -\frac{1}{1!} = -1
\]
(b) \( f(z) \) has singularities at 0 and
\[
\{\cos z = 0\} = \{z = k\pi + \pi/2 : k \in \mathbb{Z}\}.
\]
Since
\[
\lim_{z \to 0} f(z) = \left( \lim_{z \to 0} \frac{\sin z}{(\cos z)^2} \right) \left( \lim_{z \to 0} \frac{\sin z}{z} \right) = 0
\]
\( f(z) \) has a removable singularity at 0, its principal part at 0 is 0 and its residue at 0 is 0.

At \( z = k\pi + \pi/2 \) for \( k \in \mathbb{Z} \), we let \( w = z - k\pi - \pi/2 \). Then
\[
\frac{(\tan z)^2}{z} = \frac{(\cot w)^2}{w + k\pi + \frac{\pi}{2}} = \left( \frac{\cos w}{\sin w} \right)^2 \frac{1}{w + k\pi + \frac{\pi}{2}}
\]
\[
= \frac{1}{w^2} \left( \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n}}{(2n)!} \right) \left( \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n}}{(2n+1)!} \right)^{-1} \left( \sum_{n=0}^{\infty} (-1)^n \left( k\pi + \frac{\pi}{2} \right)^{n-1} w^n \right)
\]
\[
= \frac{1}{w^2} \left( \frac{2}{(2k+1)\pi} - \frac{4}{(2k+1)^2\pi^2 w} \right) + \sum_{n=0}^{\infty} a_n w^n
\]
\[
= \frac{2}{(2k+1)(z - k\pi - \pi/2)^2} - \frac{4}{(2k+1)^2\pi^2(z - k\pi - \pi/2)} + \sum_{n=0}^{\infty} a_n(z - k\pi - \pi/2)^n
\]
in \( 0 < |z - k\pi - \pi/2| < \pi \). So the principal part of \( f(z) \) at \( k\pi + \pi/2 \) is
\[
\frac{2}{(2k+1)(z - k\pi - \pi/2)^2} - \frac{4}{(2k+1)^2\pi^2(z - k\pi - \pi/2)}
\]
\( f(z) \) has a pole of order 2 at \( k\pi + \pi/2 \) and
\[
\text{Res}_{z=k\pi+\pi/2} f(z) = -\frac{4}{(2k+1)^2\pi^2}.
\]
(c) \( f(z) \) has two singularities at 0 and 1. At \( z = 0 \),
\[
\frac{e^z}{z(z-1)^2} = \frac{1}{z} \left( \sum_{n=0}^{\infty} \frac{z^n}{n!} \right) \left( \sum_{n=0}^{\infty} (n+1)z^n \right)
\]
\[
= \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n
\]
in $0 < |z| < 1$. So the principal part of $f(z)$ at 0 is

$$
\frac{1}{z}
$$

it has a pole of order 1 at 0 and

$$
\text{Res}_{z=0} f(z) = 1.
$$

At $z = 1$, we let $w = z - 1$. Then

$$
\frac{e^z}{z(z-1)^2} = \frac{e^{w+1}}{(1+w)w^2} = \frac{e^w}{w^2}(e^w(1+w)^{-1})
$$

$$
= \frac{e}{w^2} \left( \sum_{n=0}^{\infty} \frac{w^n}{n!} \right) \left( \sum_{n=0}^{\infty} (-1)^n w^n \right)
$$

$$
= \frac{e}{w^2} \left( 1 + \frac{w}{1!} + ... \right) \left( 1 - w + ... \right)
$$

$$
= \frac{e}{w^2} + \sum_{n=0}^{\infty} a_n w^n = \frac{e}{(z-1)^2} + \sum_{n=0}^{\infty} a_n (z-1)^n
$$

in $0 < |z-1| < 1$. So the principal part of $f(z)$ at 1 is

$$
\frac{e}{(z-1)^2}
$$

it has a pole of order 2 at 1 and

$$
\text{Res}_{z=1} f(z) = 0
$$

\[\square\]

(4) Let $f(z)$ be an analytic function in $D = \{|z - z_0| < R\}$. Show that

$$
\sup_{z \in D} |f(z) - f(z_0)| \geq R |f'(z_0)|.
$$

**Proof.** For all $0 < r < R$,

$$
f'(z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z) - f(z_0)}{(z-z_0)^2} dz
$$

by Cauchy Integral Formula. Therefore,

$$
|f'(z_0)| = \frac{1}{2\pi} \left| \int_{|z-z_0|=r} \frac{f(z) - f(z_0)}{(z-z_0)^2} dz \right|
$$

$$
\leq \frac{2\pi r}{2\pi} \left( \frac{1}{r^2} \right) \sup_{|z-z_0|=r} |f(z) - f(z_0)|
$$

$$
\leq \frac{2\pi r}{2\pi} \left( \frac{1}{r^2} \right) \sup_{|z-z_0|<R} |f(z) - f(z_0)|
$$
and hence
\[ \sup_{|z-z_0|<R} |f(z) - f(z_0)| \geq r|f'(z_0)| \]
for all \( r < R \). Consequently,
\[ \sup_{|z-z_0|<R} |f(z) - f(z_0)| \geq R|f'(z_0)|. \]
\( \square \)

(5) Compute the following integrals:

(a) \[ \int_\infty^{-\infty} \frac{\cos x}{2 - 2x + x^2} \, dx \]
(b) \[ \int_0^{\infty} \frac{x}{1 + x^5} \, dx \]

**Solution.**
(a) Since \( \cos x = \text{Re}(e^{ix}) \),
\[ \int_\infty^{-\infty} \frac{\cos x}{2 - 2x + x^2} \, dx = \text{Re} \left( \int_\infty^{-\infty} \frac{e^{ix}}{2 - 2x + x^2} \, dx \right) \]
is the real part of
\[ \int_\infty^{-\infty} \frac{e^{iz}}{2 - 2z + z^2} \, dz \]
For \( R > 3 \), let us consider the integral
\[ \int_{-R}^{R} \frac{e^{iz}}{2 - 2z + z^2} \, dz + \int_{C_R} \frac{e^{iz}}{2 - 2z + z^2} \, dz \]
where \( C_R \) is the semi-circle \( \{|z| = R, \text{Im}(z) \geq 0\} \).
For \( y = \text{Im}(z) \geq 0 \), \( |e^{iz}| = e^{-y} \leq 1 \). Therefore,
\[ \left| \int_{C_R} \frac{e^{iz}}{2 - 2z + z^2} \, dz \right| \leq \frac{\pi R}{R^2 - 2R - 2} \]
and hence
\[ \lim_{R \to \infty} \int_{C_R} \frac{e^{iz}}{2 - 2z + z^2} \, dz = 0. \]
Since
\[ 2 - 2z + z^2 = (z - (1 + i))(z - (1 - i)) \]
we have
\[
\int_{-R}^{R} \frac{e^{iz}}{2 - 2z + z^2} dz + \int_{C_R} \frac{e^{iz}}{2 - 2z + z^2} dz = 2\pi i \text{Res} \left( \frac{e^{iz}}{2 - 2z + z^2}, 1 + i \right) = 2\pi i \left( \frac{e^{iz}}{(2 - 2z + z^2)'} \right)_{1+i} = \pi e^{-1}(\cos(1) + i \sin(1))
\]

Taking the limit as \( R \to \infty \), we obtain
\[
\int_{-\infty}^{\infty} \frac{\cos x}{2 - 2x + x^2} dx = \lim_{R \to \infty} \text{Re} \left( \int_{-R}^{R} \frac{e^{iz}}{2 - 2z + z^2} dz \right) = \pi \cos(1) / e.
\]

(b) Let \( \alpha = \exp(2\pi i/5) \) and let us consider the complex integral

\[
(1) \quad \int_{\gamma} \frac{z}{1 + z^5} dz = \left( \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} \right) \frac{z}{1 + z^5} dz
\]
on the curve \( \gamma = \gamma_1 + \gamma_2 + \gamma_3 \) given by
\[
\begin{cases}
\gamma_1(t) = t \quad \text{for} \quad 0 \leq t \leq R \\
\gamma_2(t) = Re^{it} \quad \text{for} \quad 0 \leq t \leq 2\pi/5 \\
\gamma_3(t) = (R - t)\alpha \quad \text{for} \quad 0 \leq t \leq R
\end{cases}
\]
for some large \( R \).

For \( \gamma_2 \), we have
\[
(2) \quad \left| \int_{\gamma_2} \frac{z}{1 + z^5} dz \right| \leq \left( \frac{2\pi R}{5} \right) \frac{R}{R^5 - 1} = \frac{2\pi R^2}{5(R^5 - 1)}
\]
\[
\Rightarrow \lim_{R \to \infty} \int_{\gamma_2} \frac{z}{1 + z^5} dz = 0
\]
since \( p > q \).

For \( \gamma_1 \), we have
\[
(3) \quad \int_{\gamma_1} \frac{z}{1 + z^5} dz = \int_{0}^{R} \frac{x}{1 + x^5} dx.
\]
For \( \gamma_3 \), we have

\[
\int_{\gamma_3} \frac{z}{1 + z^5} \, dz = - \int_0^R \frac{\alpha^2(R - t)}{1 + \alpha^5(R - t)^5} \, dt
\]

(4)

\[
= -\alpha^2 \int_0^R \frac{R - t}{1 + (R - t)^5} \, dt
\]

\[
= -\alpha^2 \int_0^R \frac{x}{1 + x^5} \, dx
\]

since \( \alpha^5 = 1 \).

Combining (1)-(4), we obtain

\[
\lim_{R \to \infty} \int_{\gamma} \frac{z}{1 + z^5} \, dz = (1 - \alpha^2) \int_0^\infty \frac{x}{1 + x^5} \, dx
\]

(5)

The roots of \( 1 + z^5 \) are \( \exp((2n + 1)\pi i/5) \) for \( 0 \leq n < 5 \); among them, only \( \beta = \exp(\pi i/5) \) lies inside the curve \( \gamma \). Therefore, by Residue Theorem,

\[
\int_{\gamma} \frac{z}{1 + z^5} \, dz = 2\pi i \text{Res} \left( \frac{z}{1 + z^5}, \beta \right)
\]

(6)

\[
= 2\pi i \left. \frac{z}{(1 + z^5)'} \right|_{\beta} = \left( \frac{2\pi i}{5} \right) \beta^{-3} = -\frac{2\pi i \beta^2}{5}
\]

where \( (1 + z^5)^{-1} \) has a simple pole at \( \beta \) since \( 1 + z^5 \) has a zero at \( \beta \) of multiplicity one.

Combining (5) and (6), we obtain

\[
\int_0^\infty \frac{x}{1 + x^5} \, dx = -\left( \frac{2\pi i}{5} \right) \frac{\beta^2}{1 - \alpha^2} = -\left( \frac{2\pi i}{5} \right) \frac{\beta^2}{1 - \beta^4}
\]

\[
= -\left( \frac{2\pi i}{5} \right) \frac{1}{\beta^{-2} - \beta^2}
\]

\[
= -\left( \frac{2\pi i}{5} \right) \frac{1}{\exp(-2\pi i/5) - \exp(2\pi i/5)}
\]

\[
= -\left( \frac{2\pi i}{5} \right) \frac{1}{(-2i) \sin(2\pi/5)} = \frac{\pi}{5 \sin(2\pi/5)}
\]

where we notice that \( \alpha = \beta^2 \). \( \square \)

(6) Louville’s theorem says that there are no nonconstant bounded entire functions. Prove the same for analytic functions on \( \mathbb{C}^* \): If \( f(z) \) is analytic on \( \mathbb{C}^* \) and \( |f(z)| \leq M \) for some \( M \geq 0 \) and all \( z \neq 0 \), then \( f(z) \) is constant.
Proof. Since $|f(z)| \leq M$ for $0 < |z| < 1$, $f(z)$ has a removable singularity at 0 by Riemann Extension Theorem. Therefore, $f(z)$ extends to an entire function. By continuity,

$$|f(0)| = \lim_{z \to 0} |f(z)| \leq M.$$ 

So $|f(z)| \leq M$ for all $z \in \mathbb{C}$. By Louville’s Theorem, $f(z)$ is constant. □

(7) Let $f(z)$ and $g(z)$ be two analytic functions on the disk $\{|z| < R\}$ for some $R > 1$. Show that if $|f(z)| < |g(z)|$ for all $|z| = 1$, then either $|f(z)| < |g(z)|$ for all $|z| \leq 1$ or there exists $z_0 \in \mathbb{C}$ such that $|z_0| < 1$ and $f(z_0) = g(z_0)$.

Proof. Suppose that $g(z) \neq 0$ for all $|z| \leq 1$. Then

$$h(z) = \frac{f(z)}{g(z)}$$

is analytic on $\{|z| \leq 1\}$. Since $|f(z)| < |g(z)|$ for all $|z| = 1$, $|h(z)| < 1$ for all $|z| = 1$. Then, by maximum modulus principle,

$$|h(z)| \leq \max_{|z|=1} |h(z)| < 1$$

for all $|z| \leq 1$. Hence $|f(z)| < |g(z)|$ for all $|z| \leq 1$.

Suppose that $g(z) = 0$ for some $|z| \leq 1$. Since $|g(z)| > |f(z)|$ for all $|z| = 1$, $g(z) \neq 0$ for all $|z| = 1$. So $g(z)$ has at least one zero in $\{|z| < 1\}$. Let us apply Rouché’s Theorem to $f(z) - g(z)$ and $g(z)$: since

$$|(f(z) - g(z)) + g(z)| = |f(z)| < |g(z)|$$

$$\leq |f(z) - g(z)| + |g(z)|$$

for $|z| = 1$, $f(z) - g(z)$ and $g(z)$ have the same number of zeros in $\{|z| < 1\}$. Therefore, $f(z) - g(z)$ has at least one zero in $\{|z| < 1\}$. That is, $f(z_0) - g(z_0) = 0$, i.e., $f(z_0) = g(z_0)$ for some $|z_0| < 1$. □

(8) Let $f(z)$ be a nonconstant entire function.

(a) Show that $\overline{f(\mathbb{C})} = \mathbb{C}$, where $\overline{f(\mathbb{C})}$ is the closure of the image $f(\mathbb{C})$ of $f$.

(b) Show that both $\exp(f(z))$ and $f(e^z)$ have essential singularities at $\infty$. 
Proof. If $f(z)$ has an essential singularity at $\infty$, then
\[ f(\mathbb{C}) \supset f(\{|z| > R\}) = \mathbb{C} \]
for all $R > 0$ by Casorati-Weierstrass. Otherwise, $f(z)$ has at worst a pole at $\infty$. Then $f(z)$ is a polynomial in $z$. And since $f(z)$ is not constant, $f(z)$ is a polynomial in $z$ of degree at least 1. For every $c \in \mathbb{C}$, $f(z) - c$ has a complex root $z_0$ by Fundamental Theorem of Algebra. Therefore, $f(z_0) = c$ and $c \in f(\mathbb{C})$ for all $c \in \mathbb{C}$. Consequently, $f(\mathbb{C}) = \mathbb{C}$ for every polynomial $f(z)$ in $z$ of degree at least 1. In conclusion, $f(\mathbb{C}) = \mathbb{C}$.

If $\exp(f(z))$ has at worst a pole at $\infty$, then $\exp(f(z))$ is a polynomial in $z$. Since $f(z)$ is not constant, $f'(z) \neq 0$ and hence
\[ \frac{d}{dz} (\exp(f(z))) = f'(z) \exp(f(z)) \neq 0. \]
Therefore, $\exp(f(z))$ is a polynomial in $z$ of degree at least 1. By Fundamental Theorem of Algebra, $\exp(f(z_0)) = 0$ for some $z_0 \in \mathbb{C}$, which is impossible. Therefore, $\exp(f(z))$ has an essential singularity at 0.

If $f(\exp(z))$ has at worst a pole at $\infty$, then
\[ \lim_{z \to \infty} f(\exp(z)) = c \text{ or } \infty \]
and hence
\[ g(\{|z| > R\}) \neq \mathbb{C} \]
for some $R > 0$ and $g(z) = f(\exp(z))$. But we have proved that $f(\mathbb{C}) = \mathbb{C}$ and hence
\[ g(\{|z| > R\}) = f(\exp(\{|z| > R\})) = f(\mathbb{C}) = \mathbb{C}. \]
Contradiction. Therefore, $g(z) = f(\exp(z))$ has an essential singularity at $\infty$. \qed