Solutions for Final Review Problems

(1) Find $\sin\left(\frac{\pi}{3} + i\right)$.

**Solution.**

\[
\sin\left(\frac{\pi}{3} + i\right) = \frac{1}{2i}(e^{i(\pi/3+i)} - e^{-i(\pi/3+i)}) = \frac{1}{2i}(e^{-1}e^{\pi i/3} - e^{-\pi i/3}) = \frac{1}{2i}\left(e^{-1}(\cos\frac{\pi}{3} + i \sin \frac{\pi}{3}) - e(\cos \frac{\pi}{3} - i \sin \frac{\pi}{3})\right) = \frac{\sqrt{3}}{4}\left(e + \frac{1}{e}\right) + \frac{i}{4}\left(e - \frac{1}{e}\right)
\]

□

(2) Find the Taylor series of $(\sin z)^2$ at $z = 0$.

**Solution.** We have

\[
(\sin z)^2 = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 = -\frac{1}{4}(e^{2iz} + e^{-2iz} - 2) = -\frac{1}{4}\sum_{n=0}^{\infty} \frac{(2i)^nz^n}{n!} - \frac{1}{4}\sum_{n=0}^{\infty} \frac{(-2i)^nz^n}{n!} + \frac{1}{2}
\]

\[
= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(2i)^n z^{2n}}{(2n)!} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n-1} z^{2n}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} z^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2^{2n-1} z^{2n}}{(2n)!}
\]

□

(3) Show that

\[
|\sin(z)| \geq \sinh(|y|)
\]

for all $z \in \mathbb{C}$, where $y = \text{Im}(z)$.  

1
Proof. By triangle inequality,

\[ |\sin z| = \left| \frac{e^{iz} - e^{-2iz}}{2i} \right| \geq \frac{1}{2} \left| |e^{iz}| - |e^{-iz}| \right| \]

\[ = \frac{1}{2} \left| |e^{ix-y}| - |e^{-ix+y}| \right| = \frac{1}{2} |e^{-y} - e^y| \]

\[ = |\sinh y| = \sinh(|y|) \]

\(\square\)

(4) Let \(C_R\) denote the semicircle \(\{|z - i| = R, \text{Im}(z) \geq 1\}\). Show that

\[ \lim_{R \to \infty} \int_{C_R} \frac{dz}{z^2 \sin z} = 0 \]

Proof. When \(z \in C_R, \text{Im}(z) \geq 1\). Therefore, \(|\sin z| \geq \sinh(1)\). And since

\[ |z| = |(z - i) + i| \geq |z - i| - 1 = R - 1 \]

for \(z \in C_R, \)

\[ \left| \frac{1}{z^2 \sin z} \right| \leq \frac{1}{(R - 1)^2 \sinh(1)} \]

for \(z \in C_R\). Therefore,

\[ \left| \int_{C_R} \frac{dz}{z^2 \sin z} \right| \leq \frac{1}{(R - 1)^2 \sinh(1)} \int_{C_R} |dz| = \frac{\pi R}{(R - 1)^2 \sinh(1)} \]

Since

\[ \lim_{R \to \infty} \frac{\pi R}{(R - 1)^2 \sinh(1)} = 0 \]

we conclude

\[ \lim_{R \to \infty} \int_{C_R} \frac{dz}{z^2 \sin z} = 0 \]

\(\square\)

(5) For each of the following functions, do the following:

- find all its singularities in \(\mathbb{C}\);
- write the principal part of the function at each singularity;
- for each singularity, determine whether it is a pole, a removable singularity, or an essential singularity;
- compute the residue of the function at each singularity.

(a) \(f(z) = (1 - z) \exp \left( \frac{1}{z^2} \right)\)
(b) \( f(z) = \frac{1}{z^2 + 1} \)

(c) \( f(z) = \tan z \)

(d) \( f(z) = \frac{e^z}{z^2(z - 1)} \)

Solution. (a) \( f(z) \) has a singularity at 0. At \( z = 0 \),

\[
(1 - z) \exp \left( \frac{1}{z^2} \right) = (1 - z)e^{1/z^2} = (1 - z) \sum_{n=0}^{\infty} \frac{1}{(n!)z^{2n}}
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{(n!)z^{2n}} - \sum_{n=0}^{\infty} \frac{1}{(n!)z^{2n-1}}
\]

\[
= 1 - z + \sum_{n=1}^{\infty} \frac{1}{(n!)z^{2n}} - \sum_{n=1}^{\infty} \frac{1}{(n!)z^{2n-1}}
\]

So the principal part is

\[
\sum_{n=1}^{\infty} \frac{1}{(n!)z^{2n}} - \sum_{n=1}^{\infty} \frac{1}{(n!)z^{2n-1}}
\]

Consequently, \( f(z) \) has an essential singularity at 0 and

\[
\text{Res}_{z=0} f(z) = -\frac{1}{1!} = -1
\]

(b) \( f(z) \) has two singularities at \( \pm i \). We write

\[
\frac{1}{z^2 + 1} = \frac{1}{(z - i)(z + i)} = \frac{i}{2} \left( \frac{1}{z + i} - \frac{1}{z - i} \right)
\]

At \( i \), the principal part of \( f(z) \) is

\[
-\frac{i}{2} \frac{1}{z - i}
\]

it has a pole of order 1 and

\[
\text{Res}_{z=i} f(z) = -\frac{i}{2}
\]

At \( -i \), the principal part of \( f(z) \) is

\[
\frac{i}{2} \frac{1}{z + i}
\]

it has a pole of order 1 and

\[
\text{Res}_{z=-i} f(z) = \frac{i}{2}
\]
(c) $f(z)$ has singularities at \( \{ \cos z = 0 \} = \{ z = k\pi + \pi/2 : k \in \mathbb{Z} \} \). At $z = k\pi + \pi/2$, we let $w = z - k\pi - \pi/2$. Then

$$
\tan(z) = \tan \left( w + k\pi + \frac{\pi}{2} \right) = -\cot(w)
$$

$$
= - \frac{\cos w}{\sin w} = - \left( \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n}}{(2n)!} \right) \left( \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n+1}}{(2n+1)!} \right)^{-1}
$$

$$
= - \frac{1}{w} \left( \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n}}{(2n)!} \right) \left( \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n}}{(2n+1)!} \right)^{-1}
$$

$$
= - \frac{1}{w} \left( 1 - \frac{w^2}{2!} + \frac{w^4}{4!} - \ldots \right) \left( 1 - \frac{w^2}{3!} + \frac{w^4}{5!} - \ldots \right)^{-1}
$$

$$
= - \frac{1}{w} + \sum_{n=0}^{\infty} a_n w^n
$$

$$
= - \frac{1}{z - k\pi - \pi/2} + \sum_{n=0}^{\infty} a_n (z - k\pi - \pi/2)^n
$$

So the principal part of $f(z)$ at $k\pi + \pi/2$ is

$$
- \frac{1}{z - k\pi - \pi/2}
$$

$f(z)$ has a pole of order 1 at $k\pi + \pi/2$ and

$$
\text{Res}_{z=k\pi+\pi/2} f(z) = -1
$$

(d) $f(z)$ has two singularities at 0 and 1. At $z = 0$,

$$
\frac{e^z}{z^2(z - 1)} = - \frac{1}{z^2} \left( \sum_{n=0}^{\infty} \frac{z^n}{n!} \right) \left( \sum_{n=0}^{\infty} z^n \right)
$$

$$
= - \frac{1}{z^2} \left( 1 + \frac{z}{1!} + \ldots \right) \left( 1 + z + \ldots \right)
$$

$$
= - \frac{1}{z^2} (1 + 2z) + \sum_{n=0}^{\infty} a_n z^n = - \frac{1}{z^2} - \frac{2}{z} + \sum_{n=0}^{\infty} a_n z^n
$$

So the principal part of $f(z)$ at 0 is

$$
- \frac{1}{z^2} - \frac{2}{z}
$$

it has a pole of order 2 at 0 and

$$
\text{Res}_{z=0} f(z) = -2
$$
At $z = 1$, we let $w = z - 1$. Then
\[
\frac{e^z}{z^2(z-1)} = \frac{e^{w+1}}{(1+w)^2 w} = \frac{e^w}{w} (e^w (1+w)^{-2}) = \frac{e}{w} \left( \sum_{n=0}^{\infty} \frac{w^n}{n!} \right) \left( \sum_{n=0}^{\infty} (-1)^n(n+1)w^n \right) = \frac{e}{w} \left( 1 + \frac{w}{1!} + \ldots \right) (1-2w + \ldots) = \frac{e}{w} + \sum_{n=0}^{\infty} a_n w^n = \frac{e}{z-1} + \sum_{n=0}^{\infty} a_n (z-1)^n
\]
So the principal part of $f(z)$ at 1 is
\[
\frac{e}{z-1}
\]
it has a pole of order 1 at 1 and
\[
\text{Res}_{z=1} f(z) = e
\]

(6) Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function in a domain $D$, where $u(x,y) = \text{Re}(f(z))$ and $v(x,y) = \text{Im}(f(z))$. Show that if the function $u(x,y) + v(x,y)$ takes the maximum at a point in $D$, then $f(z)$ is constant in $D$.

Proof. We can prove the following more general statement: Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a function that does not have any local extreme. If $h(u(x,y),v(x,y))$ has a local extreme in $D$, then $f(z)$ must be constant.

Suppose that $f(z)$ is not constant and $h(u(x,y),v(x,y))$ has a local extreme at $(x_0,y_0)$. Using complex variable, we write $h(f(z)) = h(u(x,y),v(x,y))$. Then there exists $R > 0$ such that $h(f(z)) \geq h(f(z_0))$ (h($f(z)$) $\leq h(f(z_0)))$ for all $\mid z - z_0 \mid < R$.

By open mapping theorem, $f(\{\mid z - z_0 \mid < R\})$ contains a disk at $f(z_0)$. That is,
\[
\{\mid w - f(z_0) \mid < r \} \subset f(\{\mid z - z_0 \mid < R\})
\]
for some $r > 0$. Then
\[
h(w) \geq h(w_0) \ (h(w) \leq h(w_0)) \text{ for all } \mid w - w_0 \mid < r
\]
where $w_0 = f(z_0)$. This means $h(w)$ has a local extreme at $w_0$, contradicting to our hypothesis.
This proves our claim. To apply it to our situation, let $h(x, y) = x + y$. It has no local extreme since $h_x \equiv 1 \neq 0$. If $u(x, y) + v(x, y) = h(u(x, y), v(x, y))$ has a local extreme in $D$, then $f(z)$ must be constant.

(7) Let

$$f(z) = \frac{z^3}{z^2 - 3z + 2}$$

Find the Laurent series of $f(z)$ in each of the following domains:

(a) $1 < |z| < 2$;
(b) $2 < |z| < \infty$;
(c) $0 < |z - 1| < 1$.

Solution. We write $f(z)$ as a sum of partial fractions:

$$f(z) = z + 3 + \frac{8}{z - 2} - \frac{1}{z - 1}$$

Then

(a) For $1 < |z| < 2$,

$$f(z) = z + 3 + \frac{8}{z - 2} - \frac{1}{z - 1}$$

$$= z + 3 - \frac{4}{1 - (z/2)} - \frac{1}{z} \left( \frac{1}{1 - (1/z)} \right)$$

$$= z + 3 - 4 \sum_{n=0}^{\infty} 2^{-n}z^n - z^{-1} \sum_{n=0}^{\infty} z^{-n}$$

$$= z + 3 - \sum_{n=0}^{\infty} 2^{2-n}z^n - \sum_{n=0}^{\infty} z^{-n-1}$$

$$= -z - 1 - \sum_{n=2}^{\infty} 2^{2-n}z^n - \sum_{n=1}^{\infty} z^{-n}$$
(b) For $2 < |z| < \infty$,

$$f(z) = z + 3 + \frac{8}{z - 2} - \frac{1}{z - 1}$$

$$= z + 3 + \frac{8}{z} \left( \frac{1}{1 - (2/z)} \right) - \frac{1}{z} \left( \frac{1}{1 - (1/z)} \right)$$

$$= z + 3 + 8z^{-1} \sum_{n=0}^{\infty} 2^n z^{-n} - z^{-1} \sum_{n=0}^{\infty} z^{-n}$$

$$= z + 3 + \sum_{n=0}^{\infty} 2^{n+3} z^{-n-1} - \sum_{n=0}^{\infty} z^{-n-1}$$

$$= z + 3 + \sum_{n=0}^{\infty} (2^{n+3} - 1) z^{-n-1}$$

$$= z + 3 + \sum_{n=1}^{\infty} (2^{n+2} - 1) z^{-n}$$

(c) For $0 < |z - 1| < 1$,

$$f(z) = z + 3 + \frac{8}{z - 2} - \frac{1}{z - 1}$$

$$= z + 3 + \frac{8}{z - 1} - \frac{1}{z - 1}$$

$$= z + 3 - 8 \sum_{n=0}^{\infty} (z - 1)^n - \frac{1}{z - 1}$$

$$= -4 - 7(z - 1) - 8 \sum_{n=2}^{\infty} (z - 1)^n - \frac{1}{z - 1}$$

(8) Let $C$ be the circle $|z| = 1$ oriented counter-clockwise.

(a) Compute

$$\int_C \frac{z}{(4z - z^2 - 1)^2} dz$$

(b) Use or not use part (a) to compute

$$\int_0^{\pi} \frac{1}{(2 - \cos \theta)^2} d\theta$$
Solution. (a)

\[ \int_{C} \frac{z}{(4z - z^2 - 1)^2} \, dz = 2\pi i \text{Res}_{z = 2 - \sqrt{3}} \left( \frac{z}{(4z - z^2 - 1)^2} \right) \]

\[ = 2\pi i \text{Res}_{z = 2 - \sqrt{3}} \left( \frac{z - (2 - \sqrt{3})^2(z - (2 + \sqrt{3}))^2}{(z - (2 + \sqrt{3}))^2} \right) \]

\[ = 2\pi i \left. \left( \frac{z}{(z - (2 + \sqrt{3}))^2} \right)' \right|_{z = 2 - \sqrt{3}} \]

\[ = \frac{\sqrt{3}}{9} \pi \]

(b) We parameterize the circle \(|z| = 1\) with \(z = e^{i\theta}\) for \(-\pi \leq \theta \leq \pi\). Then

\[ \int_{C} \frac{z}{(4z - z^2 - 1)^2} \, dz = \int_{-\pi}^{\pi} \frac{e^{i\theta}}{(4e^{i\theta} - e^{2i\theta} - 1)^2} \, d(e^{i\theta}) \]

\[ = i \int_{-\pi}^{\pi} \frac{e^{2i\theta}}{(4e^{i\theta} - e^{2i\theta} - 1)^2} \, d\theta \]

\[ = i \int_{-\pi}^{\pi} \frac{1}{(4 - e^{i\theta} - e^{-i\theta})^2} \, d\theta \]

\[ = i \int_{-\pi}^{\pi} \frac{1}{(4 - 2\cos \theta)^2} \, d\theta = \frac{i}{4} \int_{-\pi}^{\pi} \frac{1}{(2 - \cos \theta)^2} \, d\theta \]

\[ = \frac{i}{2} \int_{0}^{\pi} \frac{1}{(2 - \cos \theta)^2} \, d\theta \]

Therefore,

\[ \int_{0}^{\pi} \frac{1}{(2 - \cos \theta)^2} \, d\theta = \frac{2\sqrt{3}}{9} \pi \]

(9) Compute the following contour integrals. You may apply Cauchy integral theorem and its corollaries wherever possible.

(a) \[ \int_{C} zdz, \]

where \(L\) is the polygonal path \(ABC\) with \(A = 0, B = 1 + i\) and \(C = 1 - i\).

(b) \[ \int_{C} z^2 \, dz \]

where \(L\) is the curve in part (a).
(c)  
\[ \int_C \frac{dz}{\sin^2 z} \]

where \( C \) is the circle \( |z| = 10 \) oriented counter-clockwise.

(d)  
\[ \int_C \frac{z}{z^{2009} + z + 1} \, dz \]

where \( C \) is the circle \( |z| = 2 \) oriented counter-clockwise.

**Solution.** (a)  
\[
\int_L zdz = \int_{AB} zdz + \int_{BC} zdz
\]
\[
= \int_0^1 (1 + i)td((1 + i)t)
\]
\[
+ \int_0^1 (1 - t)(1 + i) + t(1 - i)d((1 - t)(1 + i) + t(1 - i))
\]
\[
= (1 + i) \int_0^1 (1 - i)tdt - 2i \int_0^1 (1 - (1 - 2t)i)dt
\]
\[
= t^2 \bigg|_0^1 - 2i \left( t + \frac{i}{4}(1 - 2t)^2 \right) \bigg|_0^1 = 1 - 2i
\]

(b) Since \( z^2 \) is entire, \( z^2 \) has a complex anti-derivative \( z^3/3 \) in \( \mathbb{C} \) and
\[
\int_L z^2 dz = \frac{z^3}{3} \bigg|_0^1 = \frac{-2}{3} - \frac{2}{3}i
\]

(c) \( 1/(\sin z)^2 \) has singularities at \( k\pi \) for \( k \in \mathbb{Z} \). Hence
\[
\int_{|z|=10} \frac{dz}{(\sin z)^2} = 2\pi i \sum_{k=-3}^{3} \text{Res}_{z=k\pi} \frac{1}{(\sin z)^2}
\]
At \( z = k\pi \), we let \( w = z - k\pi \) and then

\[
\frac{1}{(\sin z)^2} = \frac{1}{(\sin(w + k\pi))^2} = \frac{1}{(\sin w)^2} \\
= \left( \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n+1}}{(2n+1)!} \right)^{-2} \\
= \frac{1}{w^2} \left( 1 - \sum_{n=1}^{\infty} \frac{(-1)^{n+1} w^{2n}}{(2n+1)!} \right)^{-2} \\
= \frac{1}{w^2} \sum_{m=0}^{\infty} (m+1) \left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1} w^{2n}}{(2n+1)!} \right)^m
\]

So the Laurent series of \( 1/(\sin w)^2 \) at \( w = 0 \) only has terms \( w^n \) with \( n \) even. Therefore,

\[
\text{Res}_{z=k\pi} \frac{1}{(\sin z)^2} = \text{Res}_{w=0} \frac{1}{(\sin w)^2} = 0
\]

Consequently,

\[
\int_{|z|=10} \frac{dz}{(\sin z)^2} = 0
\]

(d) We first show that all zeroes of \( z^{2009} + z + 1 \) lie in \( |z| < 2 \).

Otherwise, suppose that \( z^{2009} + z + 1 = 0 \) for some \( |z| \geq 2 \). Then \( 1 + z^{-2008} + z^{-2009} = 0 \). But

\[
|1 + z^{-2008} + z^{-2009}| \geq 1 - \frac{1}{|z|^{2008}} - \frac{1}{|z|^{2009}} \\
\geq 1 - \frac{1}{2^{2008}} - \frac{1}{2^{2009}} > 0
\]

for \( |z| \geq 2 \). Contradiction. So all zeroes of \( z^{2009} + z + 1 \) lie in \( |z| < 2 \). Therefore, \( z/(z^{2009} + z + 1) \) is analytic in \( |z| > 2 \). It follows that

\[
\int_C \frac{z}{z^{2009} + z + 1} = -2\pi i \text{Res}_{z=\infty} \frac{z}{z^{2009} + z + 1} \\
= 2\pi i \text{Res}_{z=0} \frac{1}{z^2} \left( \frac{z^{-1}}{z^{-2009} + z^{-1} + 1} \right) \\
= 2\pi i \text{Res}_{z=0} \frac{z^{2006}}{1 + z^{2008} + z^{2009}} = 0
\]

\(\square\)
(10) Compute the integral
\[ \int_0^\infty \frac{dx}{1 + x^r} \]
for some \( r > 1 \).

**Solution.** Let us first assume that \( r = p/q \) is rational for some positive integer \( p \) and \( q \) such that \( \gcd(p, q) = 1 \). Since \( r > 1 \), \( p > q \). Then
\[
\int_0^\infty \frac{dx}{1 + x^r} = \int_0^\infty \frac{dx}{1 + t^{p/q}} = \int_0^\infty \frac{qt^{q-1}}{1 + t^p} dt
\]
after the substitution \( x = t^q \).

Let \( \alpha = \exp(2\pi i/p) \) and let us consider the complex integral
\[
(1) \quad \int_{\gamma} \frac{qz^{q-1}}{1 + z^p} dz = \left( \int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} \right) \frac{z^{q-1}}{1 + z^p} dz
\]
along the curve \( \gamma = \gamma_1 + \gamma_2 + \gamma_3 \) given by
\[
\begin{align*}
\gamma_1(t) &= t \text{ for } 0 \leq t \leq R \\
\gamma_2(t) &= \text{Re}^{it} \text{ for } 0 \leq t \leq 2\pi/p \\
\gamma_3(t) &= (R - t)\alpha \text{ for } 0 \leq t \leq R
\end{align*}
\]
for some large \( R \).

For \( \gamma_2 \), we have
\[
\left| \int_{\gamma_2} \frac{qz^{q-1}}{1 + z^p} dz \right| \leq \left( \frac{2\pi R}{p} \right) \frac{qR^{q-1}}{R^p - 1} = \frac{2\pi R^q}{r(R^p - 1)}
\]
\[
\Rightarrow \lim_{R \to \infty} \int_{\gamma_2} \frac{qz^{q-1}}{1 + z^p} dz = 0
\]
since \( p > q \).

For \( \gamma_1 \), we have
\[
(3) \quad \int_{\gamma_1} \frac{qz^{q-1}}{1 + z^p} dz = \int_0^R \frac{qt^{q-1}}{1 + t^p} dt.
\]

For \( \gamma_3 \), we have
\[
\int_{\gamma_3} \frac{qz^{q-1}}{1 + z^p} dz = -\int_0^R \frac{q\alpha^q(R - t)^{q-1}}{1 + \alpha^p(R - t)^p} dt
\]
\[
= -\alpha^q \int_0^R \frac{q(R - t)^{q-1}}{1 + (R - t)^p} dt
\]
\[
= -\alpha^q \int_0^R \frac{qt^{q-1}}{1 + t^p} dt
\]
since $\alpha^p = 1$. Combining (1)-(4), we obtain

\begin{equation}
\lim_{R \to \infty} \int_\gamma \frac{q z^{q-1}}{1 + z^p} dz = (1 - \alpha^q) \int_0^\infty \frac{q t^{q-1}}{1 + t^p} dt
\end{equation}

The roots of $1 + z^p$ are $\exp((2n+1)\pi i/p)$ for $0 \leq n < p$; among them, only $\beta = \exp(\pi i/p)$ lies inside the curve $\gamma$. Therefore, by Residue Theorem,

\begin{equation}
\int_\gamma \frac{q z^{q-1}}{1 + z^p} dz = 2\pi i \operatorname{Res} \left( \frac{q z^{q-1}}{1 + z^p}, \beta \right)
= 2\pi i \left. \frac{q z^{q-1}}{1 + z^p} \right|_\beta
= 2\pi i \left( \frac{q}{p} \right) \beta^{q-p} = -\frac{2\pi i \beta^q}{r}
\end{equation}

where $q z^{q-1}(1 + z^p)^{-1}$ has a simple pole at $\beta$ since $1 + z^p$ has a zero at $\beta$ of multiplicity one.

Combining (5) and (6), we obtain

\begin{align*}
\int_0^\infty \frac{q t^{q-1}}{1 + t^p} dt &= -\left( \frac{2\pi i}{r} \right) \frac{\beta^q}{1 - \alpha^q} = -\left( \frac{2\pi i}{r} \right) \frac{\beta^q}{1 - \beta^{2q}} \\
&= -\left( \frac{2\pi i}{r} \right) \frac{1}{\beta - \beta^q} \\
&= -\left( \frac{2\pi i}{r} \right) \frac{1}{\exp(-q\pi i/p) - \exp(q\pi i/p)} \\
&= -\left( \frac{2\pi i}{r} \right) \frac{1}{(-2i) \sin(q\pi/p)} = \frac{\pi}{r \sin(\pi/r)}
\end{align*}

where we notice that $\alpha = \beta^2$. Therefore,

\begin{equation}
\int_0^\infty \frac{dx}{1 + x^r} = \frac{\pi}{r \sin(\pi/r)}
\end{equation}

for all rational numbers $r > 1$. It is not hard to prove that the function

$$F(r) = \int_0^\infty \frac{dx}{1 + x^r}$$

is continuous for $r > 1$. Therefore,

$$F(r) \equiv \frac{\pi}{r \sin(\pi/r)}$$

(7) holds for all real numbers $r > 1$. \qed
(11) Let $a$ be a complex number satisfying $|a| > 5/2$. Show that the power series

$$F(z) = \sum_{n=0}^{\infty} \frac{z^n}{a^{n^2}}$$

defines an entire function which does not vanish on the boundary of the annulus

$$|a^{2n-2}| < |z| < |a^{2n}|$$

and has exactly one zero inside the annulus for $n = 1, 2, \ldots$.

**Proof.** We apply Rouché’s theorem to $f(z)$ and $f_n(z) = -a^{-n^2}z^n$ in $|z| < |a^{2n}|$.

For $|z| = |a^{2n}|$,

$$\left| \frac{z^{m-1}a^{-(m-1)^2}}{z^m a^{m^2}} \right| = a^{2m-2n-1} \leq |a|^{-3} \text{ if } m \leq n - 1$$

and

$$\left| \frac{z^{m+1}a^{-(m+1)^2}}{z^m a^{m^2}} \right| = a^{2n-2m-1} \leq |a|^{-3} \text{ if } m \leq n + 1$$

Therefore,

$$|f(z) + f_n(z)| = \left| \sum_{m=0}^{n-1} a^{-m^2}z^m + \sum_{m=n+1}^{\infty} a^{-m^2}z^m \right| \leq \sum_{m=0}^{n-1} |a^{-m^2}z^m| + \sum_{m=n+1}^{\infty} |a^{-m^2}z^m|$$

$$= |a^{-(n-1)^2}z^{n-1}| \sum_{m=0}^{n-1} \left| \frac{z^m a^{-m^2}}{z^{n-1}a^{-(n-1)^2}} \right|$$

$$+ |a^{-(n+1)^2}z^{n+1}| \sum_{m=n+1}^{\infty} \left| \frac{z^m a^{-m^2}}{z^{n+1}a^{-(n+1)^2}} \right|$$

$$< |a^{n^2-1}| \sum_{m=0}^{\infty} |a|^{-3m} + |a^{n^2-1}| \sum_{m=0}^{\infty} |a|^{-3m}$$

$$= \frac{2|a|^{n^2-1}}{1 - |a|^{-3}} = |f_n(z)| \frac{2|a|^{-1}}{1 - |a|^{-3}}$$

$$= |f_n(z)| \frac{2}{|a| - |a|^{-2}} < |f_n(z)| \frac{2}{(5/2) - (5/2)^{-2}}$$

$$= \frac{100}{117} |f_n(z)| < |f_n(z)|$$
for $|z| = |a^{2n}|$ and $|a| > 5/2$. In conclusion, we have

$$|f(z) + f_n(z)| < |f(z)| + |f_n(z)|$$

for $|z| = |a^{2n}|$ and all $n = 0, 1, 2, \ldots$. By Rouché’s Theorem, $f(z)$ and $f_n(z)$ have the same number of zeros in $|z| < |a^{2n}|$, counted with multiplicity. Therefore, $f(z)$ has exactly $n$ zeros in $|z| < |a^{2n}|$, counted with multiplicity. This holds for all $n \in \mathbb{N}$.

Finally, since $f(z)$ has $n$ zeros in $|z| < |a^{2n}|$ and $n - 1$ zeros in $|z| < |a^{2n-2}|$, it has exactly one zero in $|a^{2n-2}| < |z| < |a^{2n}|$. By (9), $f(z) \neq 0$ for $|z| = |a^{2n}|$ and all $n \in \mathbb{N}$. Therefore, $f(z)$ has exactly one zero in $|a^{2n-2}| < |z| < |a^{2n}|$. □

(12) For an entire function $f(z)$, we let

$$M(r) = \max_{|z| \leq r} |f(z)|.$$ 

Let $f(z)$ be an entire function with

$$\limsup_{r \to \infty} \frac{\log M(r)}{r} = l.$$ 

Show that the infinite series

$$F(z) = \sum_{n=0}^{\infty} f^{(n)}(z)$$

converges if $l < 1$ and diverges if $l > 1$.

Proof. Suppose that $l < 1$. So there exists $\lambda < 1$ such that $|f(z)| \leq ce^{\lambda|z|}$ for some constant $c > 0$ and all $z$. By Cauchy Integral Formula,

$$|f^{(n)}(z)| = \left| \frac{n!}{2\pi i} \int_{|w|=R} \frac{f(w)}{(w-z)^{n+1}} dw \right| \leq \frac{c(n!)R}{(R-|z|)^{n+1}} e^{\lambda R}$$

for all $n \in \mathbb{N}$.

We fix $r > 0$ and want to show that

$$\sum_{n=0}^{\infty} |f^{(n)}(z)| < \infty$$

in $\{|z| \leq r\}$. We choose $R = r + \lambda^{-1}n$. Then

$$|f^{(n)}(z)| \leq ce^{\lambda r} \left( 1 + \frac{\lambda r}{n} \right) \lambda^n e^n \frac{n!}{n^n}$$
for all $n \geq 1$ and $|z| \leq r$ by (10). So it suffices to show the convergence of the series

\[
\sum_{n=1}^{\infty} c e^{\lambda r} \left( 1 + \frac{\lambda r}{n} \right) \frac{\lambda^n e^n n!}{n^n} = \sum_{n=1}^{\infty} a_n
\]

which follows from the ratio test:

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \lambda e \left( \frac{n^n}{(n+1)^n} \right) = \lambda < 1.
\]

When $l > 1$, if $F(z)$ converges for $z = 0$, then

\[
\lim_{n \to \infty} f^{(n)}(0) = 0 \Rightarrow |f^{(n)}(0)| \leq c
\]

for a constant $c$ and all $n$. Then

\[
|f(z)| = \left| \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \right| \leq \sum_{n=0}^{\infty} \frac{c|z|^n}{n!} = ce^{c|z|}
\]

which contradicts

\[
\limsup_{r \to \infty} \frac{\log M(r)}{r} = l > 1.
\]

□

(13) Let $f(z)$ be an entire function with $M(r)$ defined in the previous problem. Show that if there is a constant $0 < \alpha < 1$ such that

\[
\lim_{r \to \infty} \frac{M(\alpha r)}{M(r)} > 0,
\]

then $f(z)$ is a polynomial and the above limit is $\alpha^n$ with $n = \deg f$.

Proof. Since the limit

\[
\lim_{r \to \infty} \frac{M(\alpha r)}{M(r)} > 0,
\]

exists, there exists a constant $c > 0$ such that $M(\alpha r) \geq c M(r)$ for all $r \geq 1$. Therefore,

\[
M(\alpha^n r) \geq c^n M(r) \Rightarrow c^{-n} M(1) \geq M(\alpha^{-n}).
\]

By Cauchy Integral Formula, we have

\[
|f^{(m)}(0)| = \left| \frac{m!}{2\pi i} \int_{|z| = \alpha^{-n}} \frac{f(z)}{z^{m+1}} dz \right| \leq (m!) M(\alpha^{-n}) \alpha^{mn}
\]

\[
\leq (m!) M(1) \left( \frac{\alpha^m}{c} \right)^n
\]
for all $m$ and $n$. Then for all $m$ satisfying $\alpha^m < c$, $f^{(m)}(0) = 0$ by taking $n \to \infty$ in (20). Therefore, $f(z)$ is a polynomial.

If $f(z)$ is a polynomial of degree $n$, then

\begin{equation}
\lim_{z \to \infty} \left| \frac{f(z)}{z^n} \right| = c \Rightarrow \lim_{r \to \infty} \frac{M(r)}{r^n} = c \Rightarrow \lim_{r \to \infty} \frac{M(\alpha r)}{M(r)} = \alpha^n.
\end{equation}

\[ \square \]

(14) Suppose that the polynomial

\[ f(z) = z + a_2 z^2 + \ldots + a_n z^n \]

is one-to-one in the disk $|z| < 1$. Then $|na_n| \leq 1$.

\textbf{Proof.} Since $f(z)$ is 1-1 and analytic in $\{ |z| < 1 \}$, $f'(z) \neq 0$ for all $|z| < 1$. That is, $f'(z)$ has no zeros in $\{ |z| < 1 \}$.

If $a_n = 0$, we are done. Otherwise, let

\[ f'(z) = 1 + 2a_2 z + \ldots + na_n z^{n-1} \]

\[ = na_n (z - z_1)(z - z_2)\ldots(z - z_{n-1}) \]

where $z_1, z_2, \ldots, z_{n-1}$ are the roots of the polynomial $f'(z)$. Since $f'(z) \neq 0$ for $|z| < 1$, $|z_j| \geq 1$ for $j = 1, 2, \ldots, n - 1$. Then

\[ 1 = |f'(0)| = |na_n z_1 z_2 \ldots z_{n-1}| \geq |na_n|. \]

\[ \square \]

(15) Show that if $f : \mathbb{C} \to \mathbb{C}$ is a continuous function such that $f$ is analytic off $[0, 1]$, then $f$ is an entire function.

\textbf{Proof.} We only need that $f(z)$ is analytic on $C \setminus L$, where $L$ is the real axis $\{ y = 0 \}$.

By Goursat’s Theorem, it suffices to show that

\begin{equation}
\int_{\partial \Delta ABC} f(z) \, dz = 0
\end{equation}

for all triangles $\Delta ABC \subset \mathbb{C}$.

If $\Delta ABC \cap L = \emptyset$, then (22) follows from the analyticity of $f(z)$ in $\mathbb{C} \setminus L$.

Suppose that $\Delta ABC \subset \{ y \geq 0 \}$ or $\Delta ABC \subset \{ y \leq 0 \}$. Without loss of generality, suppose that $\Delta ABC \subset \{ y \geq 0 \}$. For $\varepsilon > 0$, let

\[ A_\varepsilon = A + \varepsilon i, \quad B_\varepsilon = B + \varepsilon i, \quad \text{and} \quad C_\varepsilon = C + \varepsilon i. \]
Then $\Delta_{AB} \subset \{y > 0\}$ and is thus disjoint from $L$. So
\[
\int_{\partial\Delta_{AB}} f(z)\,dz = 0
\]
And since
\[
\lim_{\varepsilon \to 0} \int_{\partial\Delta_{ABC}} f(z)\,dz = \int_{\partial\Delta_{ABC}} f(z)\,dz = \lim_{\varepsilon \to 0} \int_{\partial\Delta_{ADE}} f(z)\,dz + \lim_{\varepsilon \to 0} \int_{\partial\Delta_{DBE}} f(z)\,dz + \lim_{\varepsilon \to 0} \int_{\partial\Delta_{EBC}} f(z)\,dz
\]
by the continuity of $f(z)$ on $\mathbb{C}$, we conclude (22) if $\Delta ABC$ is contained in either $\{y \geq 0\}$ or $\{y \leq 0\}$.

Suppose that $\Delta ABC \not\subset \{y \geq 0\}$ and $\Delta ABC \not\subset \{y \leq 0\}$. Then the line $L$ meets the boundary of $\Delta ABC$ at two distinct points $D$ and $E$. With loss of generality, suppose that $D$ and $E$ lie on the sides $AB$ and $AC$, respectively. Then
\[
\int_{\partial\Delta_{ABC}} f(z)\,dz = \left(\int_{\partial\Delta_{ADE}} + \int_{\partial\Delta_{DBE}} + \int_{\partial\Delta_{EBC}}\right) f(z)\,dz
\]
Each of the three triangles $\Delta_{ADE}$, $\Delta_{DBE}$ and $\Delta_{EBC}$ lies entirely in either $\{y \geq 0\}$ or $\{y \leq 0\}$. So
\[
\int_{\partial\Delta_{ADE}} f(z)\,dz = \int_{\partial\Delta_{DBE}} f(z)\,dz = \int_{\partial\Delta_{EBC}} f(z)\,dz = 0
\]
and (22) follows.

(16) Let $D \subset \mathbb{C}$ be a connected open set. If $f : D \to \mathbb{C}$ is analytic except for poles, then the set of poles of $f$ has no cluster points in $D$.

Proof. Suppose that there exists a sequence of distinct points $\{z_n \in D\}$ such that $f(z)$ has poles at $z_n$ and $\lim z_n = p \in D$.

If $f(z)$ is analytic at $p$, then $f(z)$ is analytic in $\{|z - p| < r\}$ for some $r > 0$. Then
\[
\{z_n\} \cap \{|z - p| < r\} = \emptyset
\]
which contradicts $\lim z_n = p$. 


If $f(z)$ is not analytic at $p$, then $f(z)$ has a pole at $p$ by our hypothesis on $f(z)$. Then $f(z)$ is analytic in \( \{0 < |z - p| < r\} \) for some $r > 0$. Then
\[
\{z_n\} \cap \{0 < |z - p| < r\} = \emptyset
\]
which again contradicts $\lim z_n = p$ since $z_n$ are distinct.

In conclusion, the poles of $f(z)$ do not have a cluster point in $D$. \(\square\)

(17) Let $\lambda > 1$ and show that the equation $\lambda - z - e^{-x} = 0$ has exactly one solution in the half plane \( \{z : \Re z > 0\} \).

Proof. Let us apply Rouché’s Theorem to $f(z) = \lambda - z - e^{-x}$ and $g(z) = z - \lambda$ in the rectangle
\[D_n = \{0 < x < n, \ -n < y < n\}\]
for integer $n > \lambda + 1$.

We claim that
\[
|f(z) + g(z)| = |e^{-x}| < |g(z)| = |\lambda - z|
\]
for $z \in \partial D_n$, where \(\partial D_n\) is the boundary of $D_n$. Then by Rouché’s Theorem, $f(z)$ and $g(z)$ has the same number of zeros in $D_n$ and hence $f(z)$ has exactly one zero in $D_n$. This holds for all $n > \lambda + 1$. Therefore, $f(z)$ has exactly one zero in \[\bigcup D_n = \{\Re z > 0\}\].

It remains to verify (23).

When $\Re z = x = 0$,
\[
|e^{-x}| = e^{-x} = 1 < \lambda \leq \sqrt{\lambda^2 + y^2} = |\lambda - z|
\]
and (23) follows.

When $\Re z = x = n$,
\[
|e^{-x}| = e^{-n} < 1 < n - \lambda \leq |z| - \lambda < |z - \lambda|
\]
and (23) follows.

When $\Re z = x \geq 0$ and $\Im z = y = \pm n$,
\[
|e^{-x}| = e^{-x} \leq 1 < n \leq \sqrt{(x - \lambda)^2 + y^2} = |\lambda - z|
\]
and (23) follows.

This proves (23) for all $z \in \partial D_n$. \(\square\)