Math 411 Final Review

Some information on the final:

- Time and location: 14:00-16:00, Wednesday December 13, 2017, Room CAB 377.
- Materials covered by the final: Volker Runde and John C. Bowman’s Lecture Notes Chap. 1-11, 12.1, 13
- Some problems are chosen from homework (HW 1-10).

A list of topics covered by the final:

- Basics of complex numbers
- Limits and continuity of complex functions
- Complex differentiation
- Cauchy-Riemann equations and analytic functions
- Power series and radius of convergence
- Analyticity of power series
- Complex line integrals, estimate of complex line integrals and complex anti-derivatives
- Goursat’s Theorem, convex sets, star-shaped sets and simply connected sets
- Cauchy Integral Theorem and Cauchy Integral Formula
- Locally compact convergence of analytic functions
- Maximum Modulus, Open Mapping Theorem, Louville’s Theorem and Schwartz Lemma
- Biholomorphic maps, linear fractional transformation and automorphisms of $\Delta$, $\mathbb{C}$ and $\mathbb{C}^*$
- Laurent series and singularities of analytic functions
- Riemann Extension Theorem and Casorati-Weierstrass Theorem
- Winding numbers and Generalized Cauchy Integral Theorem/Formula
- Residue Theorem and its application
- Argument Principle and Rouché’s Theorem
Review Problems

(1) Find \( \sin \left( \frac{\pi}{3} + i \right) \).
(2) Find the Taylor series of \((\sin z)^2\) at \( z = 0 \).
(3) Show that \( |\sin(z)| \geq \sinh(|y|) \) for all \( z \in \mathbb{C} \), where \( y = \text{Im}(z) \).
(4) Let \( C_R \) denote the semicircle \( \{|z - i| = R, \text{Im}(z) \geq 1\} \). Show that \[
\lim_{R \to \infty} \int_{C_R} \frac{dz}{z^2 \sin z} = 0
\]
(5) For each of the following functions, do the following:
   - find all its singularites in \( \mathbb{C} \);
   - write the principal part of the function at each singularity;
   - for each singularity, determine whether it is a pole, a removable singularity, or an essential singularity;
   - compute the residue of the function at each singularity.
   - \( f(z) = (1 - z) \exp \left( \frac{1}{z^2} \right) \)
   - \( f(z) = \frac{1}{z^2 + 1} \)
   - \( f(z) = \tan z \)
   - \( f(z) = \frac{e^z}{z^2(z - 1)} \)
(6) Let \( f(z) = u(x, y) + iv(x, y) \) be an analytic function in a domain \( D \), where \( u(x, y) = \text{Re}(f(z)) \) and \( v(x, y) = \text{Im}(f(z)) \). Show that if the function \( u(x, y) + v(x, y) \) takes the maximum at a point in \( D \), then \( f(z) \) is constant in \( D \).
(7) Let \[
f(z) = \frac{z^3}{z^2 - 3z + 2}
\] Find the Laurent series of \( f(z) \) in each of the following domains:
   - \( 1 < |z| < 2 \);
   - \( 2 < |z| < \infty \);
   - \( 0 < |z - 1| < 1 \).
(8) Let \( C \) be the circle \( |z| = 1 \) oriented counter-clockwise.
(a) Compute
\[ \int_C \frac{z}{(4z - z^2 - 1)^2} \, dz \]

(b) Use or not use part (a) to compute
\[ \int_0^\pi \frac{1}{(2 - \cos \theta)^2} \, d\theta \]

(9) Compute the following contour integrals. You may apply Cauchy integral theorem and its corollaries wherever possible.
(a) \[ \int_C z \, dz, \]
where \( L \) is the polygonal path \( ABC \) with \( A = 0, B = 1 + i \) and \( C = 1 - i \).
(b) \[ \int_C z^2 \, dz \]
where \( L \) is the curve in part (a).
(c) \[ \int_C \frac{dz}{\sin^2 z} \]
where \( C \) is the circle \( |z| = 10 \) oriented counter-clockwise.
(d) \[ \int_C \frac{z}{z^{2009} + z + 1} \, dz \]
where \( C \) is the circle \( |z| = 2 \) oriented counter-clockwise.

(10) Compute the integral
\[ \int_0^\infty \frac{dx}{1 + x^r} \]
for some \( r > 1 \).

(11) Let \( a \) be a complex number satisfying \( |a| > 5/2 \). Show that the power series
\[ F(z) = \sum_{n=0}^{\infty} \frac{z^n}{a^{n^2}} \]
defines an entire function which does not vanish on the boundary of the annulus
\[ |a^{2n-2}| < |z| < |a^{2n}| \]
and has exactly one zero inside the annulus for \( n = 1, 2, \ldots \).
(12) For an entire function $f(z)$, we let 

$$M(r) = \max_{|z| \leq r} |f(z)|.$$ 

Let $f(z)$ be an entire function with 

$$\limsup_{r \to \infty} \frac{\log M(r)}{r} = l.$$ 

Show that the infinite series 

$$F(z) = \sum_{n=0}^{\infty} f^{(n)}(z)$$ 

converges if $l < 1$ and diverges if $l > 1$.

(13) Let $f(z)$ be an entire function with $M(r)$ defined in the previous problem. Show that if there is a constant $0 < \alpha < 1$ such that 

$$\lim_{r \to \infty} \frac{M(\alpha r)}{M(r)} > 0,$$ 

then $f(z)$ is a polynomial and the above limit is $\alpha^n$ with $n = \deg f$.

(14) Suppose that the polynomial 

$$f(z) = z + a_2 z^2 + \ldots + a_n z^n$$ 

is one-to-one in the disk $|z| < 1$. Then $|na_n| \leq 1$.

(15) Show that if $f : \mathbb{C} \to \mathbb{C}$ is a continuous function such that $f$ is analytic off $[0, 1]$, then $f$ is an entire function.

(16) Let $D \subset \mathbb{C}$ be a connected open set. If $f : D \to \mathbb{C}$ is analytic except for poles, then the set of poles of $f$ has no cluster points in $D$.

(17) Let $\lambda > 1$ and show that the equation $\lambda - z - e^{-z} = 0$ has exactly one solution in the half plane $\{z : \Re z > 0\}$. 