(1) No books, notes or calculators are allowed.
(2) Show your work in details.

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(1) (15 pts) Prove the Fundamental Theorem of Algebra: every complex polynomial in one variable of degree at least one has a complex root.
(2) (20 pts) Let \( f(z) \) be a nowhere vanishing analytic function on a simply connected open set \( D \). Prove the following:

(a) For every nonzero integer \( n \), there exists an analytic function \( h(z) \) such that \( (h(z))^n = f(z) \) on \( D \).
(b) If there are two analytic functions $h_1(z)$ and $h_2(z)$ on $D$ such that $(h_j(z))^n \equiv f(z)$ for $j = 1, 2$ and a nonzero integer $n$, then there exists a constant $c \in \mathbb{C}$ such that

$$c^n = 1 \text{ and } h_1(z) \equiv ch_2(z)$$

on $D$. 
(3) (30 pts) For each of the following functions, do the following:
- find all its singularities in $\mathbb{C}$;
- find the principal part of the function at each singularity;
- for each singularity, determine whether it is a pole, a removable singularity, or an essential singularity;
- compute the residue of the function at each singularity.

(a) $f(z) = (1 - z) \exp \left( \frac{1}{z^2} \right)$
(b) $f(z) = \frac{(\tan z)^2}{z}$
(c) \( f(z) = \frac{e^z}{z(z - 1)^2} \)
(4) (25 pts) Let $f(z)$ be an analytic function in $D = \{ |z - z_0| < R \}$. Show that

$$\sup_{z \in D} |f(z) - f(z_0)| \geq R |f'(z_0)|.$$
(5) (30 pts) Compute the following integrals:

(a) \[ \int_{-\infty}^{\infty} \frac{\cos x}{2 - 2x + x^2} \, dx \]
(b) \( \int_0^\infty \frac{x}{1 + x^5} dx \)
(6) (20 pts) Louville’s theorem says that there are no nonconstant bounded entire functions. Prove the same for analytic functions on $\mathbb{C}^*$: If $f(z)$ is analytic on $\mathbb{C}^*$ and $|f(z)| \leq M$ for some $M \geq 0$ and all $z \neq 0$, then $f(z)$ is constant.
(7) (30 pts) Let \( f(z) \) and \( g(z) \) be two analytic functions on the disk \( \{ |z| < R \} \) for some \( R > 1 \). Show that if \( |f(z)| < |g(z)| \) for all \( |z| = 1 \), then either \( |f(z)| < |g(z)| \) for all \( |z| \leq 1 \) or there exists \( z_0 \in \mathbb{C} \) such that \( |z_0| < 1 \) and \( f(z_0) = g(z_0) \).
(8) (30+20 pts) Let \( f(z) \) be a nonconstant entire function.

(a) (30 pts) Show that \( \overline{f(\mathbb{C})} = \mathbb{C} \), where \( \overline{f(\mathbb{C})} \) is the closure of the image \( f(\mathbb{C}) \) of \( f \).
(b) (Bonus +20 pts) Show that both \( \exp(f(z)) \) and \( f(e^z) \) have essential singularities at \( \infty \).
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