Solution for Midterm I [1]

1. (a) The level curves of $f(x,y)$ are ellipses $x^2/c^2 + y^2/(4c^2) = 1$ with the short axis c (along x-axis) and the long axis $2c$ (along y-axis) for $0 < c \leq 4$.

   (b) $\partial f/\partial x = -4x/\sqrt{4-4x^2-y^2}$ and $\partial f/\partial y = -y/\sqrt{4-4x^2-y^2}$.

   (c) The tangent plane is $z - f(1/2,1) = f_x(1/2,1)(x-1/2) + f_y(1/2,1)(y-1)$, i.e., $z - \sqrt{2} = -\sqrt{2}(x-1/2) - \sqrt{2}(y-1)/2$.

2. (30 points) (a) (b) The equation has two equilibrium solutions $p = 0$ and $p = M$. If $p > M$, $p' < 0$; if $M > p > 0$, $p' > 0$; if $p < 0$, $p' < 0$. So $p = 0$ is unstable and $p = M$ is stable.

   (c) By separation of variables,

$$\int \frac{1}{p(M-p)} dp = \int k dt.$$

   The LHS can be integrated using partial fractions:

$$\begin{aligned}\int \frac{1}{p(M-p)} dp &= \int \frac{1}{M}\left(\frac{1}{p} + \frac{1}{M-p}\right) dp \\ &= \frac{1}{M}\left(\ln|p| - \ln|M-p|\right) + C = \frac{1}{M}\ln\left|\frac{p}{M-p}\right| + C\end{aligned}$$

   Therefore,

$$\frac{1}{M}\ln\left|\frac{p}{M-p}\right| = kt + C \Rightarrow \frac{p}{M-p} = Ae^{kMt}.$$

   Solve it for p and we obtain

$$p = \frac{AMe^{kMt}}{1+Ae^{kMt}} = \frac{AM}{A+e^{-kMt}}.$$

   (c) Given the information, $p(0) = M/3$ and $p(100) = M/2$. The former tells us that

$$\frac{AM}{1+A} = \frac{M}{3}$$

[1] http://www.math.ucsb.edu/~xichen/math3c02s/mid1sol.pdf

and hence $A = 1/2$. The latter tells us

$$\frac{AM}{A + e^{-100kM}} = M/2$$

and hence $e^{-100kM} = 1/2$. Therefore, $e^{-200kM} = 1/4$ and

$$p(200) = \frac{AM}{A + e^{-200kM}} = \frac{2}{3}M$$

The limit is obviously $\lim_{t\to\infty} p(t) = M$ since $\lim_{t\to\infty} e^{-kMt} = 0$.

3. (a) $y_1 = y_0 + h(y_0 - 1) = 7/3$, $y_2 = y_1 + h(y_1 - 1) = 25/9$, $y_3 = y_2 + h(y_2 - 1) = 91/27$. So $y(1) \approx 91/27$.

   (b) The general solution of the equation is $y = 1 + Ae^t$. With $y(0) = 2$, we get $A = 1$. So the solution is $y(t) = 1 + e^t$ and hence $y(1) = 1 + e$. The error is $|(91/27) - (1 + e)| = e - (64/27)$.

4. (a) By separation of variables,

$$\int e^{-y} dy = \int xe^{x^2} dx$$

   Use the substitution $u = x^2$ for the RHS. So

$$-e^{-y} + C = \frac{1}{2}e^{x^2} \Rightarrow y = \ln\left(C - \frac{1}{2}e^{x^2}\right).$$

   (b) By separation of variables,

$$\int \frac{1}{y+1} dy = \int \frac{1}{t+1} dt \Rightarrow \ln|y+1| = \ln|t+1| + C.$$

   Therefore $y + 1 = A(t + 1)$. With $y(0) = 2$, we obtain $A = 3$. The solution is $y = 3t + 2$.

   (c) Change it to the form $(t - 1)y' = ty$. By separation of variables,

$$\int \frac{1}{y} dy = \int \frac{t}{t-1} dt.$$

   The RHS is

$$\int \frac{t}{t-1} dt = \int \left(1 + \frac{1}{t-1}\right) dt = t + \ln|t-1| + C.$$

   The solution is

$$\ln|y| = t + \ln|t-1| + C \Rightarrow y = A(t-1)e^t.$$