

PRINT NAME: \_\_\_\_\_

STUDENT ID NUMBER: \_\_\_\_\_

- (1) No books and notes are allowed.
- (2) You may use a scientific calculator and a notecard.
- (3) Write clearly and show your work in details.

Problem	Points	Score
1	20	
2	30	
3	20	
4	30	
Total	100	

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(1) (20 points) Let  $f(x, y) = \sqrt{4 - 4x^2 - y^2}$ .

(a) (5 points) Sketch the level curves of  $f(x, y)$  (draw at least three level curves).

(b) (10 points) Compute the partial derivatives  $\partial f/\partial x$  and  $\partial f/\partial y$  of  $f$ .

(c) (5 points) Find the tangent plane of  $z = f(x, y)$  at the point  $(x, y) = (1/2, 1)$ .

- (2) (30 points) Let  $p(t)$  be the population of a country at time  $t$  with  $t$  measured in years. Suppose that  $p(t)$  satisfies the differential equation

$$\frac{dp}{dt} = k(M - p)p$$

for some positive constants  $k$  and  $M$ .

- (a) (5 points) Sketch the direction field of the equation.

- (b) (5 points) Find the equilibrium solutions of the equation and determine their stabilities.

(c) (10 points) Find the general solution of the equation.

(d) (10 points) Suppose that the population is  $M/3$  in year zero and the population is  $M/2$  in year 100. What is the population in year 200? What is the limit of the population as  $t$  goes to infinity? (Your answers should only involve  $M$  not  $k$ .)

(3) (20 points) Let  $y(t)$  be the solution of

$$\frac{dy}{dt} = y - 1$$

satisfying the initial condition  $y(0) = 2$ .

(a) (10 points) Approximate  $y(1)$  using Euler's method with step-size  $h = 1/3$ .

(b) (10 points) What is the error in the approximation of part (a)?

(4) (30 points) Solve the following differential equations.

(a)  $\frac{dy}{dx} = xe^{x^2+y}$

(b)  $\frac{dy}{dt} = \frac{y+1}{t+1}$  with  $y(0) = 2$

(c)  $ty' - y' = ty$