Solution for the Final

1. (a) \(\frac{\partial f}{\partial x} = e^{xy} + xy e^{xy} + ye^x\) and \(\frac{\partial f}{\partial y} = x^2 e^{xy} + e^x\).

   (b) The tangent plane is \(z - 2e = 3e(x - 1) + 2e(y - 1)\).

2. (a) \(y_0 = 1, \ x_0 = 1, \ x_n = x_0 + nh\) and \(y_{n+1} = y_n + (x_n + y_n)h\). So
\(y_1 = 1 + (1 + 1)(2/3) = 7/3, \ y_2 = 7/3 + (5/3 + 7/3)(2/3) = 5, \ y_3 = 5 + (7/3 + 5)(2/3) = 89/9\).

   (b) Solve the equation using the method of integrating factor:
\[e^{-x}y = \int xe^{-x}dx = -\int xd(e^{-x})\]
\[= \int e^{-x}dx - xe^{-x} = -xe^{-x} - e^{-x} + C\]
\[y = -x - 1 + Ce^x.\]

   The initial condition \(y(1) = 1\) gives \(C = 3e^{-1}\). So the solution is \(y = -x - 1 + 3e^{x-1}\).

   (c) The real value of \(y(3)\) is \(-3 - 1 + 3e^2 = 3e^2 - 4\). So the absolute error is \(|3e^2 - 4 - 89/9| = 3e^2 - 125/9\).

3. (a) Solve \(y - 2y^3 = 0 \Rightarrow y(1 - 2y^2) = 0\). So the equilibrium solutions are \(y = 0, \ y = \sqrt{2}/2\) and \(y = -\sqrt{2}/2\).

   (b) When \(y > \sqrt{2}/2, \ y' < 0; \) when \(0 < y < \sqrt{2}/2, \ y' > 0; \) when \(-\sqrt{2}/2 < y < 0, \ y' < 0; \) when \(y < -\sqrt{2}/2, \ y' > 0\). So \(y = \pm\sqrt{2}/2\) are stable and \(y = 0\) is unstable.

   (c) \(\lim_{t \to \infty} y(t) = -\sqrt{2}/2\) from the direction field.

4. (a) Separation of variables:
\[\int \ln y dy = \int \ln x dx \Rightarrow y \ln y - y = x \ln x - x + C\]

   (b) Using method of integrating factor:
\[y' + e^x y = e^{2x} \Rightarrow e^{x} (y' + e^x y) = e^{2x} \Rightarrow e^{x} y = \int e^{2x} e^{x} dx.\]

\[1\text{http://www.math.ucsb.edu/~xichen/math3c02s/fsol.pdf}\]
To work out the integral on the right, we make a substitution $u = e^x$:

$$\int e^{2x} e^x \, dx = \int u \, du = u^2 - u + C.$$ 

So the general solution is

$$y = e^x - 1 + C e^{-e^x}.$$ 

Use the initial condition $y(0) = 1$ to find $C$ and we obtain $C = e$. So the solution is

$$y = e^x - 1 + e^{1-e^x}.$$ 

(c) The integrating factor is

$$I(t) = e^{\int (2+1/t) \, dt} = e^{2t+\ln t} = te^{2t}.$$ 

So

$$te^{2t} y = \int 2te^{2t} = \int td(e^{2t})$$

$$= te^{2t} - \int e^{2t} \, dt = te^{2t} - \frac{e^{2t}}{2} + C.$$ 

So the solution is

$$y = 1 - \frac{1}{2t} + \frac{C}{t e^{2t}}.$$ 

(d) Separation of variables:

$$xy' = xy - y \Rightarrow xy' = (x - 1)y$$

$$\Rightarrow \int \frac{1}{y} \, dy = \int \frac{x - 1}{x} \, dx \Rightarrow \ln |y| = x - \ln |x| + C$$

$$\Rightarrow xy = Ae^x.$$ 

The initial condition $y(2) = 1$ gives us $A = 2e^{-2}$. So the solution is

$$xy = 2e^{x-2}.$$ 

5. Since $f'(x) = \sec^2 x$ and $f''(x) = (\sec^2 x)' = 2 \sec x (\sec x \tan x) = 2 \sec^2 x \tan x$, $f'(\pi/4) = 2$ and $f''(\pi/4) = 4$. The second Taylor polynomial is

$$1 + 2 \left(x - \frac{\pi}{4}\right) + 2 \left(x - \frac{\pi}{4}\right)^2$$

2
6. (a) Since
\[ \frac{1}{1 + x} = \sum (-1)^n x^n \]
substitute \( x \) by \( x^2 \) and we obtain
\[ \frac{1}{1 + x^2} = \sum (-1)^n x^{2n}. \]
So
\[ \frac{1 + x}{1 + x^2} = \sum (-1)^n x^{2n} + \sum (-1)^n x^{2n+1}. \]
(b) Note that \( 2002 = 2 \cdot 1001 \) and \((-1)^{1001} = -1 \). So the coefficient is \(-1\).

7. (a) Let \( y(t) \) be the balance of your account after \( t \) month, \( r \) be the monthly interest rate and \( m \) be your monthly payment. Then the differential equation modeling this situation is \( dy/dt = ry - m + 100 \). The general solution is \( y = Ae^{rt} + (m - 100)/r \). Since \( y(0) = 2400 \), \( A = 2400 - (m - 100)/r \). So the solution is
\[ y = (2400 - (m - 100)/r)e^{rt} + (m - 100)/r \]
In two years (24 months), \( y(24) = 0 \). So
\[ (2400 - (m - 100)/r)e^{24r} + (m - 100)/r = 0 \]
Solve it for \( m \) and we have
\[ m = 100 + \frac{2400re^{24r}}{e^{24r} - 1} \]
Plug in \( r = 0.01 \) and we have \( m = 212.48 \).
(b) The total interest paid is \( 24m - 2400 - 2400 = 299.51 \).

8. Let \( x(t) \) be the amount of salt in the tank at time \( t \). Then
\[ \frac{dx}{dt} = -3 \left( \frac{x}{100} \right) \]
with \( x(0) = 20 \). The solution of this IVP is \( x(t) = 20e^{-0.03t} \). We want to find \( t \) such that \( x(t) = 10 \). So we solve the equation \( 20e^{-0.03t} = 10 \) and obtain \( t = 100(ln 2)/3 = 23.1 \). So after 23.1 minutes, the tank contains the correct amount of salt.
9. Let \( y(t) \) be the population (in the unit of \( 10^9 \) individuals) after \( t \) hours. Then the logistic equation is
\[
\frac{dy}{dt} = ky \left( 1 - \frac{y}{M} \right),
\]
where \( M = 5 \), \( y(0) = 1 \) and \( y'(0) = 1 \). Hence
\[
y'(0) = ky(0) \left( 1 - \frac{y(0)}{M} \right).
\]
So \( k = 5/4 \).

The solution of the equation is
\[
y(t) = \frac{5}{1 + 4e^{-kt}}.
\]
So after 4 hours,
\[
y(4) = \frac{5}{1 + 4e^{-4k}} = \frac{5}{1 + e^{-5}}.
\]