Solution for Midterm II [1]

1. Let $y(t)$ be your credit card balance (in dollar), t be the time variable (in month), $r = 0.12/12 = 0.01$ be the interest rate and m be your monthly payment (in dollar). Then the differential equation modelling this situation is

$$\frac{dy}{dt} = ry - m + 100$$

with $y(0) = 2400$ and $y(24) = 0$. The general solution of the equation is

$$y(t) = Ae^{rt} + \frac{m-100}{r}.$$

Use the initial condition $y(0) = 2400$ to obtain A:

$$2400 = A + \frac{m-100}{r} \Rightarrow A = 2400 - \frac{m-100}{r}.$$

Therefore

$$y(t) = \left(2400 - \frac{m-100}{r}\right)e^{rt} + \frac{m-100}{r}.$$

The condition $y(24) = 0$ translates to

$$0 = \left(2400 - \frac{m-100}{r}\right)e^{24r} + \frac{m-100}{r}.$$

Solve it for m and we have

$$\frac{m-100}{r} = \frac{2400e^{24r}}{e^{24r}-1} \Rightarrow m = 100 + \frac{2400re^{24r}}{e^{24r}-1}.$$

Evaluate m by plugging in $r = 0.01$: $m = 212$.

The total interest is: $24m - (2400 + 24 \cdot 100) = 300$. Alternatively, one can compute the total interest by evaluating $\int_0^{24} ry(t)dt$.

[1] http://www.math.ucsb.edu/~xichen/math3c01w/mid2sol.pdf

2. (a) The integrating factor is $I(x) = e^{e^x}$. After multiplying both sides by $I(x)$ and then integrating, we have

$$\begin{aligned} e^{e^x} y &= \int e^{e^x} e^{2x} dx \text{ (Substitue } u = e^x) \\ &= \int e^u u^2 \left(\frac{1}{u} du \right) \\ &= \int u e^u du = u e^u - e^u + C = e^x e^{e^x} - e^{e^x} + C. \end{aligned}$$

Hence the general solution is

$$y = e^x - 1 + C e^{-e^x}.$$

The initial condition $y(0) = 1$ gives $C = e$. So the solution is

$$y = e^x - 1 + e^{1-e^x}.$$

(b) The integrating factor is $I(x) = e^{3x}$. After multiplying both sides by $I(x)$ and then integrating, we have

$$\begin{aligned} e^{3x} y &= \int x e^{3x} dx + \int e^{5x} dx \\ &= \frac{1}{3} \int x d e^{3x} + \frac{1}{5} e^{5x} \\ &= \frac{1}{3} x e^{3x} - \frac{1}{3} \int e^{3x} dx + \frac{1}{5} e^{5x} \\ &= \frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} + \frac{1}{5} e^{5x} + C. \end{aligned}$$

So the general solution is

$$y = \frac{1}{3} x - \frac{1}{9} + \frac{1}{5} e^{2x} + C e^{-3x}.$$

3. (a) The integrating factor is $I(t) = e^{kt}$. After multiplying both sides

by $I(t)$ and then integrating, we have

$$\begin{aligned} e^{kt}x &= \int te^{kt}dt \\ &= \frac{1}{k}\int tde^{kt} \\ &= \frac{1}{k}te^{kt} - \frac{1}{k}\int e^{kt}dt \\ &= \frac{1}{k}te^{kt} - \frac{1}{k^2}e^{kt} + C. \end{aligned}$$

So the general solution is

$$x = \frac{1}{k}t - \frac{1}{k^2} + Ce^{-kt}.$$

The initial condition $x(0) = x_0$ gives $C = k^{-2} + x_0$. So the solution is

$$x = \frac{1}{k}t - \frac{1}{k^2} + \left(x_0 + \frac{1}{k^2}\right)e^{-kt}.$$

(b) Replacing x_0 in part (a) by $x_0 = 1$ and $\hat{x}_0 = 2$ respectively, we have

$$x(t) = \frac{1}{k}t - \frac{1}{k^2} + \left(1 + \frac{1}{k^2}\right)e^{-kt}$$

and

$$\hat{x}(t) = \frac{1}{k}t - \frac{1}{k^2} + \left(2 + \frac{1}{k^2}\right)e^{-kt}.$$

So the absolute error is $E_a = |x(t) - \hat{x}(t)| = e^{-kt}$ and the relative error is

$$E_r = \frac{E_a}{|x(t)|} = \frac{e^{-kt}}{|k^{-1}t - k^{-2} + (1 + k^{-2})\,e^{-kt}|}.$$

(c) In order for $\lim_{t\to\infty} e^{-kt} = 0$ to hold, we must have $k > 0$ since otherwise if $k < 0$, $\lim_{t\to\infty} e^{-kt} = \infty$.