Solution for Midterm I [1]

1. For part (a), solve the equation by separation of variables:

$$\int \frac{1}{e^p - 1} dp = \int dt.$$

The integral on the left hand side can be worked out by substituting e^p, which results in a rational function that can be integrated using partial fractions.

Substituting $u = e^p$ ($du = e^p dp$ and hence $dp = (1/u)du$):

$$\int \frac{1}{e^p - 1} dp = \int \frac{1}{u(u-1)} du.$$

Writing $1/(u(u-1))$ as a sum of partial fractions:

$$\frac{1}{u(u-1)} = \frac{1}{u-1} - \frac{1}{u}.$$

Hence

$$\begin{aligned} \int \frac{1}{e^p - 1} dp &= \int \frac{1}{u(u-1)} du \\ &= \int \frac{1}{u-1} du - \int \frac{1}{u} du \\ &= \ln \left| \frac{u-1}{u} \right| = \ln \left| \frac{e^p - 1}{e^p} \right|. \end{aligned}$$

Therefore, the solution to the equation is

$$\ln \left| \frac{e^p - 1}{e^p} \right| = t + C \Rightarrow \frac{e^p - 1}{e^p} = Ae^t.$$

Expressing p in terms of t:

$$e^p = \frac{1}{1 - Ae^t} \Rightarrow p = -\ln(1 - Ae^t).$$

[1] http://www.math.ucsb.edu/~xichen/math3c01w/mid1sol.pdf

For part (b), using initial condition $p(0) = -1$ to find A: $-1 = -\ln(1-A) \Rightarrow A = 1-e$. So the solution satisfying $p(0) = -1$ is $p(t) = -\ln(1+(e-1)e^t)$ and $\lim_{t\to\infty} p(t) = -\infty$.

For part (c), the only equilibrium solution is $p = 0$ by solving $e^p - 1 = 0$. It is unstable since $f'(p) = (e^p - 1)' = e^p$ and $f'(0) = 1 > 0$.

Some comments: most students failed to get part (a) despite the fact that I gave a very similar problem in the sample midterm. The partial credits are given as follows: for (a), 2 points for separation of variables, 2 points for substitution, 2 points for partial fractions, 4 points for the rest; for (b), as you need the result in (a), you won't get any credit if your answer in (a) is far off; for (c), 5 points for the equilibrium solution and 5 points for stability.

2. For part (a), $y_1 = y_0 + f(t_0, y_0)h = 1 + (1+1)\cdot 1 = 3$ and $y_2 = y_1 + f(t_1, y_1)h = 3 + (3+1)\cdot 1 = 7$. The approximation for $y(3)$ is $y_2 = 7$.

   For part (b), solve the equation by separation of variables: $y = Ae^t - 1$. The initial condition $y(1) = 1$ yields $A = 2/e$. So the solution is $y(t) = (2/e)e^t - 1 = 2e^{t-1} - 1$.

   For part (c), the real value of $y(3)$ is $y(3) = 2e^2 - 1$ by (b). So the error is $|2e^2 - 8|$.

   Some comments: there is nothing tricky about this problem and the computation is minimal. Some students messed up the number of iterations needed (only two iterations here and they came up with three). The partial credits are given as follows: for (a), only minor points are deducted if you make some numerical mistakes; for (b), 1 point for separation of variables, 2 points for the general solution, 2 points for using initial condition; for (c), no points are deducted if you get the answer wrong because you get (b) or (c) wrong, but you have to show me that you understand what is going on.

3. For part (a), $f_x = \partial f/\partial x = y\sec^2(xy)$ and $f_y = \partial f/\partial y = x\sec^2(xy)$.

   For part (b), $f_x(1, \pi/4) = \pi/2$, $f_y(1, \pi/4) = 2$ and the tangent plane at $(1, \pi/4, 1)$ is $z - 1 = (\pi/2)(x-1) + 2(y - \pi/4)$.

   Some comments: most students got this one. The partial credits are given as follows: for (a), 5 points for each derivative; for (b), no points will be deducted if you get the wrong answer because you get the derivatives wrong in (a), but you have to show me that you know how to find the tangent plane.

4. For (a), first factorize $y - xy = y(1 - x)$ and then solve it by separation of variables:

$$\int \frac{1}{y} dy = \int (1 - x) dx \Rightarrow \ln |y| = x - x^2/2 + C.$$

So the general solution is $y = Ae^{x - x^2/2}$.

For (b), solve it by separation of variables:

$$\int \frac{1}{e^y} dy = \int dx.$$

The LHS is

$$\int \frac{1}{e^y} dy = \int e^{-y} dy = -e^{-y}.$$

So the general solution is

$$-e^{-y} = x + C \Rightarrow y = -\ln(-x - C).$$

Use the initial condition $y(0) = 1$ to obtain $-e^{-1} = C$. So the solution is $y = -\ln(e^{-1} - x)$.

Some comments: most students got (a). Some students had trouble integrating $1/e^y$. I want to emphasize that $\int (1/e^y) dy$ is definitely not $\ln(e^y)$. Many students did this integral the hard way by substituting e^y although they did finally get the right answer. Also many students did the following: from $-e^{-y} = x + C$ they obtained $y = -\ln|-x - C|$. Where this $|\cdot|$ comes from really puzzled me; it is entirely unnecessary and, strictly speaking, wrong in this case, although I did not take any points off this time. The partial credits are given as follows: for (a), 2 points for separation of variables and 8 points for the rest; for (b), 2 points for separation of variables, 6 points for the general solution and 2 points for using the initial condition.

5. Let $y = C$ be the constant solution. Then $0 = C(1 - x)$ for all x. So C must be zero. The only constant solution is $y = 0$.

   Another way to do it is to use the general solution $y = Ae^{x-x^2/2}$ obtained in the last problem. The only constant solution is $y = 0$ when $A = 0$.

   Some comments: I've got all kinds of (right/wrong) answers and it's hard to summarize here.