Solution for Midterm II [1]

1. (a) Solve the equation using the method of integrating factor:

$$e^{3t}x = \int te^{3t}dt = \frac{1}{3}\int tde^{3t} = \frac{1}{3}\left(te^{3t} - \int e^{3t}dt\right)$$
$$= \frac{1}{3}\left(te^{3t} + \frac{1}{3}e^{3t}\right) + C = \frac{1}{3}te^{3t} - \frac{1}{9}e^{3t} + C$$
$$\Rightarrow x = \frac{1}{3}t - \frac{1}{9} + Ce^{-3t}.$$

By the initial condition $x(0) = 0$, we have $0 = C - 1/9$ so $C = 1/9$. The solution is

$$x = \frac{1}{3}t - \frac{1}{9} + \frac{1}{9}e^{-3t}$$

where $t/3 - 1/9$ is the steady-state part and $e^{-3t}/9$ is the transient part.

(b) Solve the equation using the method of integrating factor after dividing both sides of the equation by t^4:

$$tx = \int t\left(\frac{1}{t^4}\right)dt = -\frac{1}{2t^2} + C$$
$$\Rightarrow x = -\frac{1}{2t^3} + \frac{C}{t}.$$

2. Let $m(t)$ be the mass of the sample. Then $m(t) = m_0e^{kt}$. Since it takes n days for it to decay from $10g$ to $9g$, $10e^{nk} = 9 \Rightarrow k = n^{-1}\ln(9/10)$. To find the time it takes for the sample to decay from $9g$ to $8g$, it suffices to solve $9e^{kt} = 8$ for t. So it takes

$$t = \frac{1}{k}\ln\frac{8}{9} = n\left(\frac{\ln(8/9)}{\ln(9/10)}\right)$$

days for it to decay from $9g$ to $8g$.

3. (a) Let $y(t)$ be the money you owe (in dollars) after t months, r be the monthly interest rate and m be the monthly payments. The differential equation is $dy/dt = ry - m$. Solve it by separation of variables and

[1] http://www.math.ucsb.edu/~xichen/math3c01s/mid2webbsol.pdf

we obtain $y = Ae^{rt} + m/r$. By the initial condition $y(0) = 15000$, $A = 15000 - m/r$. So the solution is $y = (15000 - m/r)e^{rt} + m/r$. After 30 years (360 months), $y(360) = 0$. So we have the equation $(15000 - m/r)e^{360r} + m/r = 0$. Solve it for m and we have

$$m = \frac{15000re^{360r}}{e^{360r} - 1}.$$

Take $r = 0.09/12$ and we have $m = 120.6$. So the monthly payments are 120.6 dollars.

(b) The total interest in 30 years is $360m - 15000 = 28418$ dollars.

4. (a) Solve the equation using the method of integrating factor:

$$e^{2t}x = \int mte^{2t}dt = \frac{m}{2}te^{2t} - \frac{m}{4}e^{2t} + C$$
$$\Rightarrow x = \frac{mt}{2} - \frac{m}{4} + Ce^{-2t}.$$

By the initial condition $x(0) = 1$, $C - m/4 = 1$ and hence $C = 1 + m/4$. So the solution is

$$x = \frac{mt}{2} - \frac{m}{4} + (1 + \frac{m}{4})e^{-2t}.$$

(b) If $m = 1$, $x(t) = t/2 - 1/4 + 5e^{-2t}/4$. If $\hat{m} = 2$, $\hat{x}(t) = t - 1/2 + 3e^{-2t}/2$. So the errors are

$$E_a = \left|\frac{t}{2} - \frac{1}{4} + \frac{e^{-2t}}{4}\right| \text{ and } E_r = \left|\frac{2t - 1 + e^{-2t}}{2t - 1 + 5e^{-2t}}\right|.$$

(c) For E_a, we see that $t/2 - 1/4 \to \infty$ and $e^{-2t}/4 \to 0$ as $t \to \infty$. Therefore $\lim_{t\to\infty} E_a = \infty$.

For E_r, we observe that the dominant term for the enumerator is $2t$ while the dominant term for the denominator is $2t$, as $t \to \infty$. So $\lim_{t\to\infty} E_r = \lim_{t\to\infty} 2t/(2t) = 1$. However, the rigorous way to put this is to use L'Hôspital:

$$\lim_{t\to\infty} E_r = \lim_{t\to\infty} \left|\frac{2t - 1 + e^{-2t}}{2t - 1 + 5e^{-2t}}\right|$$
$$= \lim_{t\to\infty} \left|\frac{(2t - 1 + e^{-2t})'}{(2t - 1 + 5e^{-2t})'}\right| = \lim_{t\to\infty} \left|\frac{2 - 2e^{-2t}}{2 - 10e^{-2t}}\right| = 1.$$