Solution for Midterm II

1. (a) Solve the equation using the method of integrating factor:

\[ e^{3t}x = \int t e^{3t} dt = \frac{1}{3} \int tde^{3t} = \frac{1}{3} \left( te^{3t} - \int e^{3t} dt \right) \]

\[ = \frac{1}{3} \left( te^{3t} + \frac{1}{3} e^{3t} \right) + C = \frac{1}{3} te^{3t} - \frac{1}{9} e^{3t} + C \]

\[ \Rightarrow x = \frac{1}{3} t - \frac{1}{9} + Ce^{-3t}. \]

By the initial condition \( x(0) = 0 \), we have \( 0 = C - 1/9 \) so \( C = 1/9 \).

The solution is

\[ x = \frac{1}{3} t - \frac{1}{9} + \frac{1}{9} e^{-3t} \]

where \( t/3 - 1/9 \) is the steady-state part and \( e^{-3t}/9 \) is the transient part.

(b) Solve the equation using the method of integrating factor after dividing both sides of the equation by \( t^4 \):

\[ tx = \int t \left( \frac{1}{t^4} \right) dt = -\frac{1}{2t^2} + C \]

\[ \Rightarrow x = -\frac{1}{2t^2} + \frac{C}{t}. \]

2. Let \( m(t) \) be the mass of the sample. Then \( m(t) = m_0e^{kt} \). Since it takes \( n \) days for it to decay from 10g to 9g, \( 10e^{nk} = 9 \Rightarrow k = n^{-1} \ln(9/10) \).

To find the time it takes for the sample to decay from 9g to 8g, it suffices to solve \( 9e^{kt} = 8 \) for \( t \). So it takes

\[ t = \frac{1}{k} \ln \frac{8}{9} = n \left( \frac{\ln(8/9)}{\ln(9/10)} \right) \]

days for it to decay from 9g to 8g.

3. (a) Let \( y(t) \) be the money you owe (in dollars) after \( t \) months, \( r \) be the monthly interest rate and \( m \) be the monthly payments. The differential equation is \( dy/dt = ry - m \). Solve it by separation of variables and

1http://www.math.ucsb.edu/~xichen/math3c01s/mid2webbsol.pdf
we obtain \( y = Ae^{rt} + m/r \). By the initial condition \( y(0) = 15000 \), 
\( A = 15000 - m/r \). So the solution is \( y = (15000 - m/r)e^{rt} + m/r \). 
After 30 years (360 months), \( y(360) = 0 \). So we have the equation 
\( (15000 - m/r)e^{360r} + m/r = 0 \). Solve it for \( m \) and we have 
\[
m = \frac{15000re^{360r}}{e^{360r} - 1}.
\]
Take \( r = 0.09/12 \) and we have \( m = 120.6 \). So the monthly payments 
are 120.6 dollars.

(b) The total interest in 30 years is \( 360m - 15000 = 28418 \) dollars.

4. (a) Solve the equation using the method of integrating factor:
\[
e^{2t}x = \int mte^{2t}dt = \frac{m}{2}te^{2t} - \frac{m}{4}e^{2t} + C
\]
\[
\Rightarrow x = \frac{mt}{2} - \frac{m}{4} + Ce^{-2t}.
\]
By the initial condition \( x(0) = 1 \), \( C - m/4 = 1 \) and hence \( C = 1 + m/4 \). 
So the solution is 
\[
x = \frac{mt}{2} - \frac{m}{4} + (1 + \frac{m}{4})e^{-2t}.
\]

(b) If \( m = 1 \), \( x(t) = t/2 - 1/4 + 5e^{-2t}/4 \). If \( m = 2 \), \( \dot{x}(t) = t - 1/2 + 3e^{-2t}/2 \). So the errors are
\[
E_a = \left| \frac{t}{2} - \frac{1}{4} + \frac{e^{-2t}}{4} \right| 
\text{and} 
E_r = \left| \frac{2t - 1 + e^{-2t}}{2t - 1 + 5e^{-2t}} \right|.
\]

(c) For \( E_a \), we see that \( t/2 - 1/4 \to \infty \) and \( e^{-2t}/4 \to 0 \) as \( t \to \infty \). 
Therefore \( \lim_{t \to \infty} E_a = \infty \).

For \( E_r \), we observe that the dominant term for the enumerator is \( 2t \) 
while the dominant term for the denominator is \( 2t \), as \( t \to \infty \). So 
\( \lim_{t \to \infty} E_r = \lim_{t \to \infty} 2t/(2t) = 1 \). However, the rigorous way to put 
this is to use L'Hôpital:
\[
\lim_{t \to \infty} E_r = \lim_{t \to \infty} \left| \frac{2t - 1 + e^{-2t}}{2t - 1 + 5e^{-2t}} \right| 
= \lim_{t \to \infty} \left| \frac{(2t - 1 + e^{-2t})'}{(2t - 1 + 5e^{-2t})'} \right| 
= \lim_{t \to \infty} \left| \frac{2 - 2e^{-2t}}{2 - 10e^{-2t}} \right| = 1.
\]