Solution for Midterm II [1]

1. (a) Solve the equation using the method of integrating factor:

$$\begin{aligned} e^{-2t}x &= \int te^{-2t}dt = -\frac{1}{2}\int tde^{-2t} = -\frac{1}{2}\left(te^{-2t} - \int e^{-2t}dt\right) \\ &= -\frac{1}{2}\left(te^{-2t} + \frac{1}{2}e^{-2t}\right) + C = -\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} + C \\ \Rightarrow x &= -\frac{1}{2}t - \frac{1}{4} + Ce^{2t}. \end{aligned}$$

   By the initial condition $x(0) = 0$, we have $0 = C - 1/4$ so $C = 1/4$. The solution is

$$x = -\frac{1}{2}t - \frac{1}{4} + \frac{1}{4}e^{2t}.$$

   (b) Solve the equation using the method of integrating factor after dividing both sides of the equation by t^5:

$$\begin{aligned} tx &= \int t\left(\frac{1}{t^5}\right)dt = -\frac{1}{3t^3} + C \\ \Rightarrow x &= -\frac{1}{3t^4} + \frac{C}{t}. \end{aligned}$$

2. Let $m(t)$ be the mass of the sample. Then $m(t) = m_0e^{kt}$. Since it takes n days for it to decay from $9g$ to $8g$, $9e^{nk} = 8 \Rightarrow k = n^{-1}\ln(8/9)$. To find the time it takes for the sample to decay from $8g$ to $7g$, it suffices to solve $8e^{kt} = 7$ for t. So it takes

$$t = \frac{1}{k}\ln\frac{7}{8} = n\left(\frac{\ln(7/8)}{\ln(8/9)}\right)$$

   days for it to decay from $8g$ to $7g$.

3. (a) Let $y(t)$ be the money you owe (in dollars) after t months, r be the monthly interest rate and m be the monthly payments. The differential equation is $dy/dt = ry - m$. Solve it by separation of variables and we obtain $y = Ae^{rt} + m/r$. By the initial condition $y(0) = 5000$,

[1] http://www.math.ucsb.edu/~xichen/math3c01s/mid2pyschsol.pdf

$A = 5000 - m/r$. So the solution is $y = (5000 - m/r)e^{rt} + m/r$. After 2 years (24 months), $y(24) = 0$. So we have the equation $(5000 - m/r)e^{24r} + m/r = 0$. Solve it for m and we have

$$m = \frac{5000re^{24r}}{e^{24r} - 1}.$$

Take $r = 0.06/12$ and we have $m = 221$. So the monthly payments are 221 dollars.

(b) The total interest in two years is $24m - 5000 = 304$ dollars.

4. (a) Solve the equation using the method of integrating factor:

$$e^{kt}x = \int e^{(k+1)t}dt = \frac{1}{k+1}e^{(k+1)t} + C$$
$$\Rightarrow x = \frac{1}{k+1}e^{t} + Ce^{-kt}.$$

By the initial condition $x(0) = 1$, $1/(k+1) + C = 1$ and hence $C = k/(k+1)$. So the solution is

$$x = \frac{1}{k+1}e^{t} + \frac{k}{k+1}e^{-kt}.$$

(b) If $k = 1$, $x(t) = e^{t}/2 + e^{-t}/2$. If $\hat{k} = 2$, $\hat{x}(t) = e^{t}/3 + 2e^{-2t}/3$. So the errors are

$$E_a = \left|\frac{e^{t}}{6} + \frac{e^{-t}}{2} - \frac{2}{3}e^{-2t}\right| \text{ and } E_r = \left|\frac{e^{t} + 3e^{-t} - 4e^{-2t}}{3e^{t} + 3e^{-t}}\right|.$$

(c) For E_a, we see that $e^{t}/6 \to \infty$, $e^{-t}/2 \to 0$ and $2e^{-2t}/3 \to 0$ as $t \to \infty$. Therefore $\lim_{t\to\infty} E_a = \infty$.

For E_r, we observe that the dominant term for the enumerator is e^{t} while the dominant term for the denominator is $3e^{t}$, as $t \to \infty$. So $\lim_{t\to\infty} E_r = \lim_{t\to\infty} e^{t}/(3e^{t}) = 1/3$. However, the rigorous way to put this is

$$\lim_{t\to\infty} E_r = \lim_{t\to\infty}\left|\frac{e^{t} + 3e^{-t} - 4e^{-2t}}{3e^{t} + 3e^{-t}}\right|$$
$$= \lim_{t\to\infty}\left|\frac{(e^{t} + 3e^{-t} - 4e^{-2t})/e^{t}}{(3e^{t} + 3e^{-t})/e^{t}}\right| = \lim_{t\to\infty}\left|\frac{1 + 3e^{-2t} - 4e^{-3t}}{3 + 3e^{-2t}}\right| = \frac{1}{3}.$$