# Solution for Homework 4 [1]

**[KA] Sec. 1.5, page 74**

**1.** For any values of $x_0$, the solution is $x = x_0 e^{kt}$.

**4.** No. The function $f(t, x) = x^2 - t^2$ cannot be written as the product $f(t, x) = g(t)h(x)$. The product $(x + t)(x - t)$ is not in the form $g(t)h(x)$.

Although this is not required, I will show you a rigorous way to prove $x' = x^2 - t^2$ is not separable as follows. If $f(t, x) = g(t)h(x)$, then $f(1, x)/f(0, x) = g(1)/g(0)$ is a constant; however, $f(1, x)/f(0, x) = (x^2 - 1)/x^2$ is not a constant.

**6.** Solve the equation:

$$
\begin{aligned}
\frac{dp}{dt} = p(1-p) \Rightarrow \int \frac{1}{p(1-p)} dp = \int dt \Rightarrow \int \frac{1}{p} dp + \int \frac{1}{1-p} dp = t + C \\
\Rightarrow \ln|p| - \ln|1-p| = t + C \Rightarrow \ln\left|\frac{p}{1-p}\right| = t + C \\
\Rightarrow \frac{p}{1-p} = Ae^t \Rightarrow p = Ae^t(1-p) \\
\Rightarrow p + (Ae^t)p = Ae^t \Rightarrow p = \frac{Ae^t}{1 + Ae^t}.
\end{aligned}
$$

If $0 < p(0) = p_0 < 1$, $p' > 0$, which implies $p(t)$ is increasing. So $\lim_{t\to\infty} p(t)$ exists and $p(t)$ approaches an equilibrium solution as $t \to \infty$. This equilibrium solution must be $p = 1$. So $\lim_{t\to\infty} p(t) = 1$.

If $p(0) = p_0 > 1$, $p' < 0$, which implies $p(t)$ is decreasing. So $\lim_{t\to\infty} p(t)$ exists and $p(t)$ approaches an equilibrium solution as $t \to \infty$. This equilibrium solution must be $p = 1$. So $\lim_{t\to\infty} p(t) = 1$.

Alternatively, the limit $\lim_{t\to\infty} p(t)$ can be computed using the general solution obtained in part (a). The general solution of the equation is $p =$

[1] http://www.math.ucsb.edu/~xichen/math3c01s/hw4key.pdf

$Ae^t/(1 + Ae^t)$. So as long as $A \neq 0$,

$$\begin{aligned}\lim_{t\to\infty} p(t) &= \lim_{t\to\infty} \frac{Ae^t}{1 + Ae^t} \\ &= \lim_{t\to\infty} \frac{A}{e^{-t} + A} = \frac{A}{\lim_{t\to\infty} e^{-t} + A} = 1.\end{aligned}$$

**7.** Equilibrium solutions: (a) $x = -1$ (b) None (c) $x = 0$ (d) None.

Separable: (a) Yes (b) Yes (c) Yes (d) No.

(a) The general solution is $x = Ae^t - 1$. The solution satisfying $x(0) = 1$ is $x = 2e^t - 1$ and the solution satisfying $x(0) = -1$ is $x = -1$.

(b) The general solution is $x = \tan(t+C)$ ($\int (x^2+1)^{-1} dx = \tan^{-1} x + C$). The solution satisfying $x(0) = 1$ is $x = \tan(t + \pi/4)$ and the solution satisfying $x(0) = -1$ is $x = \tan(t - \pi/4)$.

(c) The general solution is $x = (C - t)^2/4$. The solution satisfying $x(0) = 1$ is $x = (2 - t)^2/4$. No solutions satisfy $x(0) = -1$.

**8.** Equilibrium solutions: (a) $x = 0$ (b) None (c) None (d) None (e) $x = 1$

Separable: (a) No (b) Yes (c) Yes (d) Yes (e) Yes

(b) The general solution is $x^2 = t^2 + C$. The solution satisfying $x(0) = 1$ is $x = \sqrt{t^2 + 1}$ and the solution satisfying $x(0) = -1$ is $x = -\sqrt{t^2 + 1}$.

(c) The general solution is $e^x = e^t + C$. The solution satisfying $x(0) = 1$ is $x = \ln(e^t + e - 1)$ and the solution satisfying $x(0) = -1$ is $x = \ln(e^t + e^{-1} - 1)$.

(d) The general solution is $\int e^{-x^2} dx = t^2/2$. The solution satisfying $x(0) = 1$ is $\int_1^x e^{-u^2} du = t^2/2$. The solution satisfying $x(0) = -1$ is $\int_{-1}^x e^{-u^2} du = t^2/2$.

(e) The general solution is $\int (\ln x)^{-1} dx = t$. The solution satisfying $x(0) = 1$ is $x = 1$. No solutions satisfy $x(0) = -1$.

**9.** (a) $\sec x + \tan x = Ae^t$ ($\int \sec x dx = \ln|\sec x + \tan x| + C$)

(b) Since $\sec x + \tan x = \tan(x/2 + \pi/4)$, $x = 2\tan^{-1}(Ae^t) - \pi/2 + 2k\pi$ where $k$ is an integer.

(c) If $x(0) = 0$, $A = 1$. So the solution is $\sec x + \tan x = e^t$.

(d) $x = \pi/2$ since $x = \pi/2$ is an equilibrium solution.

(e) From the direction field, $\lim_{t\to\infty} x(t) = \pi/2$ in both (c) and (d).

**11.** (a) (b) (See **6**) $x = Ae^t/(1 + Ae^t)$.

(c) If $x(0) = 1/2$, $A = 1$. So the solution is $x(t) = e^t/(1 + e^t)$.

(d) If $x(0) = 2$, $A = -2$. So the solution is $x(t) = 2e^t/(2e^t - 1)$.

(e) (See **6**) Both limits are 1.

**12.** Write $(p(p-1)(2-p))^{-1}$ as partial fractions:

$$\frac{1}{p(p-1)(2-p)} = -\frac{1}{2}\left(\frac{1}{p}\right) + \frac{1}{p-1} + \frac{1}{2}\left(\frac{1}{2-p}\right).$$

So the solution is

$$\begin{aligned}
&\int \frac{1}{p(p-1)(2-p)} dp = t + C \\
&\Rightarrow -\frac{1}{2}\int \frac{1}{p} dp + \int \frac{1}{p-1} dp + \frac{1}{2}\int \frac{1}{2-p} dp = t + C \\
&\Rightarrow -\frac{1}{2}\ln|p| + \ln|p-1| - \frac{1}{2}\ln|2-p| = t + C \\
&\Rightarrow -\ln|p| + 2\ln|p-1| - \ln|2-p| = 2t + C \\
&\Rightarrow \ln\left|\frac{(p-1)^2}{p(2-p)}\right| = 2t + C \\
&\Rightarrow (p-1)^2 = Ae^{2t}p(2-p).
\end{aligned}$$

If $p(0) = 1/2$ or $p(0) = 3/2$, $A = 1/3$. So the solution is $3(p-1)^2 = e^{2t}p(2-p)$. Solve it as a quadratic equation for $p$:

$$\begin{aligned}
&(3 + e^{2t})p^2 - (6 + 2e^{2t})p + 3 = 0 \Rightarrow \\
p &= \frac{(3 + e^{2t}) \pm \sqrt{(3 + e^{2t})^2 - 3(3 + e^{2t})}}{3 + e^{2t}} \\
&= 1 \pm \sqrt{1 - \frac{3}{3 + e^{2t}}} = 1 \pm \frac{e^t}{\sqrt{3 + e^{2t}}}
\end{aligned}$$

So the solution satisfying $p(0) = 1/2$ is

$$p = 1 - \frac{e^t}{\sqrt{3 + e^{2t}}}$$

and the solution satisfying $p(0) = 3/2$ is

$$p = 1 + \frac{e^t}{\sqrt{3 + e^{2t}}}.$$