Sample Midterm I

(1) (30 points) Consider the following differential equation
\[ \frac{dp}{dt} = 1 - e^p. \]
(a) (10 points) Find the general solution.
(b) (5 points) Let \( p(0) = p_0 \). Suppose that \( p_0 < 0 \). Find \( \lim_{t \to \infty} p(t) \).
(c) (5 points) Let \( p(0) = p_0 \). Suppose that \( p_0 > 0 \). Find \( \lim_{t \to \infty} p(t) \).
(d) (10 points) Find all the equilibrium solutions of the equation and determine their stabilities.

(2) (20 points) Consider the following differential equation
\[ \frac{dy}{dt} = y + t \]
with initial condition \( y(1) = 0 \).
(a) (10 points) Use Euler’s method to approximate \( y(2) \) by taking the step \( h = 1/3 \).
(b) (5 points) Verify that \( y(t) = Ae^t - t - 1 \) is the general solution of the equation, where \( A \) is a constant.
(c) (5 points) What is the absolute error in the approximation of part (a)?

(3) (20 points) Let \( f(x, y) = x^2 - xy + 2y^2 \).
(a) (10 points) Compute \( \partial f / \partial x \) and \( \partial f / \partial y \).
(b) (10 points) Find the tangent plane of \( z = f(x, y) \) at the point \( (1, 1, 2) \).

(4) (20 points) Solve the following differential equations.
(a) (10 points)
\[ \frac{dy}{dx} = xy. \]
(b) (10 points)
\[ \frac{dy}{dx} = y + 1 \]
with initial condition \( y(0) = 0 \).
(5) (10 points) Find all the constant solutions of the differential equation

\[ \frac{dy}{dx} = x^2 + y^2. \]

You must justify your answer.
(1) (a) 

\[ p = \ln \frac{Ae^t}{1 + Ae^t}. \]

(b) 0
(c) 0
(d) \( p = 0 \). Stable.

(2) (a) \( 47/27 \).
(c) \( 2e^3 - 47/27 \).

(3) (a) \( \frac{\partial f}{\partial x} = 2x - y \) and \( \frac{\partial f}{\partial y} = -x + 4y \).
(b) \( z - 2 = (x - 1) + 3(y - 1) \).

(4) (a) \( y = Ae^{x^2/2} \).
(b) \( y = e^x - 1 \).

(5) No constant solution.