

## Solution for Practice Final <sup>1</sup>

(1)

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{3}{n} \sqrt{1 + \frac{3i}{n}} = \int_0^3 \sqrt{1+x} dx.$$

(2) Since  $1 \leq \sqrt{1+x^{100}} \leq \sqrt{2}$  for  $-1 \leq x \leq 1$ ,

$$\int_{-1}^1 1 dx \leq \int_{-1}^1 \sqrt{1+x^{100}} dx \leq \int_{-1}^1 \sqrt{2} dx$$

i.e.,

$$2 \leq \int_{-1}^1 \sqrt{1+x^{100}} dx \leq 2\sqrt{2}.$$

(3) Let  $G(t)$  be the antiderivative of  $\cos(t^2)$ . Then by Fundamental Theorem of Calculus,  $F(x) = \int_x^2 \cos(t^2) dt = G(2) - G(x)$ . So  $F'(x) = (G(2) - G(x))' = -G'(x) = -\cos(x^2)$ .

(4) (a) Plug  $x = 6t^2$  and  $y = t^3 - 12t$  to each side of the  $216y^2 = x(x - 72)^2$  and show they are the same.

(b) The point  $(72, 0)$  corresponds to both  $t = 2\sqrt{3}$  and  $t = -2\sqrt{3}$ . So it cross itself at  $(72, 0)$ .  $dy/dx = \pm\sqrt{3}/3$  at  $t = \pm 2\sqrt{3}$ . So the tangent lines are  $y = \pm\sqrt{3}/3(x - 72)$ .

(c)  $dy/dx = (3t^2 - 12)/(12t)$ . So it has two horizontal tangent lines at  $t = \pm 2$ , which are  $y = \pm 16$ . It has a vertical tangent line at  $t = 0$ , which is  $x = 0$ .

(d)

$$\int_0^1 \sqrt{(12t)^2 + (3t^2 - 12)^2} dt = \int_0^1 3(t^2 + 4) dt = 13$$

(e)

$$2\pi \int_0^1 6t^2 \sqrt{(12t)^2 + (3t^2 - 12)^2} dt = 2\pi \int_0^1 18t^2(t^2 + 4) dt = \frac{276}{5}\pi$$

(f)

$$\int_0^{-1} (t^3 - 12t) d(6t^2) = \int_0^{-1} 12t(t^3 - 12t) dt = \frac{228}{5}$$

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<sup>1</sup><http://www.math.ucsb.edu/~xichen/math3b00w/pfsol.pdf>

- (5) (a) Substitute  $u = \tan x$ , then integrate by parts and finally integrate a rational function by long division

$$\begin{aligned} \int \sec^2 x \ln(1 + \tan x) dx &= \int \ln(1 + \tan x) d \tan x \\ &= \int \ln(1 + u) du = u \ln(1 + u) - \int u d \ln(1 + u) \\ &= u \ln(1 + u) - \int \frac{u}{1 + u} du = u \ln(1 + u) - \int \left(1 - \frac{1}{1 + u}\right) du \\ &= u \ln(1 + u) - u + \ln(1 + u) + C \\ &= \tan x \ln(1 + \tan x) - \tan x + \ln(1 + \tan x) + C \end{aligned}$$

- (b) Integrate by parts

$$\begin{aligned} \int_0^{\pi/3} x \cos x dx &= \int_0^{\pi/3} x d \sin x \\ &= x \sin x \Big|_0^{\pi/3} - \int_0^{\pi/3} \sin x dx = \frac{\sqrt{3}}{6} \pi - \frac{1}{2} \end{aligned}$$

- (c) Integrate by parts

$$\begin{aligned} \int x^2 \ln x dx &= \frac{1}{3} \int \ln x dx^3 = \frac{1}{3} (x^3 \ln x - \int x^3 d \ln x) \\ &= \frac{1}{3} x^3 \ln x - \frac{1}{3} \int x^2 dx = \frac{1}{3} x^3 \ln x - \frac{x^3}{9} \end{aligned}$$

- (d) Substitute  $u = \tan x$

$$\begin{aligned} \int \sec^4 x dx &= \int \sec^2 x d \tan x = \int (u^2 + 1) du \\ &= \frac{u^3}{3} + u + C = \frac{\tan^3 x}{3} + \tan x + C \end{aligned}$$

- (e) Integrate a rational function by long division and partial fractions

$$\begin{aligned} \int \frac{x^2 + 2x - 4}{x^2 + 3x - 4} dx &= \int \left(1 - \frac{x}{x^2 + 3x - 4}\right) dx \\ &= x - \int \frac{x}{(x + 4)(x - 1)} dx \\ &= x - \frac{4}{5} \int \frac{dx}{x + 4} - \frac{1}{5} \int \frac{dx}{x - 1} \\ &= x - \frac{4}{5} \ln |x + 4| - \frac{1}{5} \ln |x - 1| + C \end{aligned}$$

(6) By the second integral we are given,

$$\begin{aligned}
 \int \tan^6 x dx &= \frac{\tan^5 x}{5} - \int \tan^4 x dx \\
 &= \frac{\tan^5 x}{5} - \left( \frac{\tan^3 x}{3} - \int \tan^2 x dx \right) \\
 &= \frac{1}{5} \tan^5 x - \frac{1}{3} \tan^3 x + (\tan x - \int 1 dx) \\
 &= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + C
 \end{aligned}$$

Integrate by parts

$$\begin{aligned}
 \int x \tan^6 x dx &= \int x d \left( \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x \right) \\
 &= x \left( \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x \right) \\
 &\quad - \int \left( \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x \right) dx.
 \end{aligned}$$

By the second integral given,

$$\int \tan^3 dx = \frac{\tan^2 x}{2} - \int \tan x dx = \frac{\tan^2 x}{2} + \ln |\cos x| + C$$

and

$$\int \tan^5 dx = \frac{\tan^4}{4} - \int \tan^3 x dx = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} - \ln |\cos x| + C$$

Finally,

$$\begin{aligned}
 \int x \tan^6 x dx &= x \left( \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - \frac{x}{2} \right) \\
 &\quad - \left( \frac{\tan^4 x}{20} - \frac{4}{15} \tan^2 x - \frac{23}{15} \ln |\cos x| \right) + C
 \end{aligned}$$

(7)

$$\pi \int_0^1 ((2-x^2)^2 - (2-x)^2) dx = \frac{8}{15} \pi$$

(8)

$$\int_{-r}^r (2\sqrt{r^2-x^2})^2 dx = \frac{4}{3} r^3$$

$$(9) \quad dy/dx = x - \frac{1}{4x}$$

$$\begin{aligned} \int_1^5 \sqrt{1 + \left(x - \frac{1}{4x}\right)^2} dx &= \int_1^5 \sqrt{1 + x^2 - \frac{1}{2} + \frac{1}{16x^2}} dx \\ &= \int_1^5 \sqrt{x^2 + \frac{1}{2} + \frac{1}{16x^2}} dx \\ &= \int_1^5 \sqrt{\left(x + \frac{1}{4x}\right)^2} dx \\ &= \int_1^5 \left(x + \frac{1}{4x}\right) dx = 12 + \frac{\ln 5}{4} \end{aligned}$$

(10) Since  $|(1/x)''| = 2x^{-3} \leq 1/4$  for  $2 \leq x \leq 4$ , we take  $K = 1/4$ .  
Then

$$\left| \int_2^4 \frac{1}{x} dx - M_n \right| \leq \frac{(1/4)(4-2)^3}{24n^2} = \frac{1}{12n^2}.$$

To make  $1/(12n^2) \leq 10^{-3}$ ,  $n \geq \sqrt{10^3/12} \Rightarrow n \geq 10$ . So we should take  $n = 10$ .

(11) Divergent since

$$\frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} > \frac{\sqrt{\sqrt{x}}}{\sqrt{x}} = \frac{1}{x^{1/4}}$$

and

$$\begin{aligned} \int_1^\infty x^{-1/4} dx &= \lim_{t \rightarrow \infty} \left[ \frac{4}{3} x^{3/4} \right]_1^t \\ &= \lim_{t \rightarrow \infty} \frac{4}{3} (t^{3/4} - 1) = \infty \end{aligned}$$

(12)

$$\begin{aligned} 2\pi \int_{-1}^2 y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx &= 2\pi \int_{-1}^2 2\sqrt{x+1} \sqrt{1 + \left(\frac{1}{\sqrt{x+1}}\right)^2} dx \\ &= 4\pi \int_{-1}^2 \sqrt{x+2} dx \\ &= 4\pi \left[ \frac{2}{3} (x+2)^{3/2} \right]_{-1}^2 = \frac{56}{3}\pi \end{aligned}$$

(13) Solve  $r = \cos \theta$  and  $r = \sin \theta \Rightarrow \cos \theta = \sin \theta \Rightarrow \tan \theta = 1 \Rightarrow \theta = \pi/4$ . So the intersection is  $(\sqrt{2}/2, \pi/4)$ .

The area is

$$\begin{aligned} & \frac{1}{2} \int_0^{\pi/4} \sin^2 \theta d\theta + \frac{1}{2} \int_{\pi/4}^{\pi/2} \cos^2 \theta d\theta \\ &= \frac{1}{2} \int_0^{\pi/4} \frac{1}{2}(1 - \cos 2\theta) d\theta + \frac{1}{2} \int_{\pi/4}^{\pi/2} \frac{1}{2}(1 + \cos 2\theta) d\theta \\ &= \frac{\pi}{8} - \frac{1}{4} \end{aligned}$$

(14)

$$\begin{aligned} \int_0^{2\pi} \sqrt{(\theta^2)^2 + (2\theta)^2} d\theta &= \int_0^{2\pi} \theta \sqrt{\theta^2 + 4} d\theta \\ &= \frac{1}{2} \int_0^{4\pi^2} \sqrt{u + 4} du = \frac{8}{3} ((\pi^2 + 1)^{3/2} - 1) \end{aligned}$$

(15)

$$\int_0^{10} \frac{60}{40}(40 - x) dx = 525 \text{ ft-lb}$$