Solution for Sample Midterm II

(1) (a) \((\pi/2)(\sin(\pi/4 + \pi/4) + \sin(3\pi/4 + \pi/4) + \sin(5\pi/4 + \pi/4) + \sin(7\pi/4 + \pi/4)) = 0\)

(b) \((\pi/4)(\sin(\pi/4)+2\sin(\pi/2+\pi/4)+2\sin(\pi+\pi/4)+2\sin(3\pi/2+\pi/4) + \sin(2\pi + \pi/4)) = 0\)

(c) \((\pi/6)(\sin(\pi/4)+4\sin(\pi/2+\pi/4)+2\sin(\pi+\pi/4)+4\sin(3\pi/2+\pi/4) + \sin(2\pi + \pi/4)) = 0\)

(2) (a) Convergent since \(\sin x \leq x\) for \(x \geq 0\) \(\Rightarrow 0 \leq x^{-3/2}\sin x \leq x^{-1/2}\) and

\[\int_0^1 x^{-1/2} \, dx = 2 \left[ x^{1/2} \right]_0^1 = 2\]

is convergent.

(b) Convergent since \(x^2 + e^x \geq e^x \Rightarrow 0 \leq \frac{1}{x^2 + e^x} \leq e^{-x}\) and

\[\int_1^\infty e^{-x} \, dx = -e^{-x}]_1^\infty = e^{-1}\]

is convergent.

(c) Convergent since \(\sqrt{x^4 + 1} \geq x^2 \Rightarrow 0 \leq \frac{1}{\sqrt{x^4 + 1}} \leq \frac{1}{x^2}\) and

\[\int_1^\infty \frac{dx}{x^2} = -\frac{1}{x}]_1^\infty = 1\]

is convergent.

(3) (a) Use the method of cylindrical shell

\[2\pi \int_0^1 ye^y \, dy = 2\pi(ye^y - e^y)]_0^1 = 2\pi\]

(b) \[\pi \int_0^1 (e^y)^2 \, dy = \frac{\pi}{2}e^{2y}]_0^1 = \frac{\pi}{2}(e^2 - 1)\]

(c) \[(1000)(10)\pi \int_0^1 (e^y)^2(1 - y) \, dy = 5000\pi \int_0^1 (1 - y)e^{2y} \, dy\]

\[= 5000\pi \left( (1 - y)e^{2y}]_0^1 + \int_0^1 e^{2y} \, dy \right) = 2500\pi(e^2 - 3) \, J\]

\(^1\)http://www.math.ucsb.edu/~xichen/math3b00w/p2sol.pdf
(4) (a) Substitute \( u = e^t \) and then integrate a rational function by expressing it as a sum of partial fractions

\[
\int \frac{e^t - 1}{e^t + 1} dt = \int \frac{e^t - 1}{e^t(e^t + 1)} de^t \\
= \int \frac{u - 1}{u(u + 1)} du = \int \frac{2}{u + 1} du - \int \frac{1}{u} du \\
= 2 \ln |u + 1| - \ln |u| + C = 2 \ln(e^t + 1) - t + C
\]

(b) Use the trigonometric identity \( \sin A \sin B = \frac{1}{2} \cos(A - B) - \frac{1}{2} \cos(A + B) \)

\[
\int_{0}^{\pi/4} (\sin 5x)(\sin 2x)dx = \frac{1}{2} \int_{0}^{\pi/4} (\cos 3x - \cos 7x)dx \\
= \frac{1}{2} \left( \frac{\sin 3x}{3} - \frac{\sin 7x}{7} \right)_{0}^{\pi/4} = \frac{5\sqrt{2}}{42}
\]

(c) Substitute \( u = \sqrt{x^2 + 3} \) \( (du = x(x^2 + 3)^{-1/2}dx) \) and then integrate a rational function by expressing it as a sum of partial fractions

\[
\int \frac{dx}{x\sqrt{x^2 + 3}} = \int \frac{1}{u^2 - 3} du = \frac{1}{2\sqrt{3}} \int \left( \frac{1}{u - \sqrt{3}} - \frac{1}{u + \sqrt{3}} \right) du \\
= \frac{\sqrt{3}}{6} \ln |u - \sqrt{3}| - \frac{\sqrt{3}}{6} \ln |u + \sqrt{3}| + C \\
= \frac{\sqrt{3}}{6} \ln(\sqrt{x^2 + 3} - \sqrt{3}) - \frac{\sqrt{3}}{6} \ln(\sqrt{x^2 + 3} + \sqrt{3}) + C
\]

(d) Integrate by parts

\[
\int x^3 e^x dx = \int x^3 e^x dx = x^3 e^x - 3 \int x^2 e^x dx \\
= x^3 e^x - 3 \int x^2 e^x = x^3 e^x - 3(\int x^2 e^x - 2 \int xe^x dx) \\
= x^3 e^x - 3x^2 e^x + 6 \int x e^x = x^3 e^x - 3x^2 e^x + 6(xe^x - \int e^x dx) \\
= x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + C
\]
(e) Integrate a rational function by expressing it as a sum of partial fractions

\[
\int \frac{1}{x^4 - 1} \, dx = \int \frac{1}{(x - 1)(x + 1)(x^2 + 1)} \, dx
\]

\[
= \int \left( \frac{1}{4} \cdot \frac{1}{x - 1} - \frac{1}{4} \cdot \frac{1}{x + 1} - \frac{1}{2} \cdot \frac{1}{x^2 + 1} \right) \, dx
\]

\[
= \frac{1}{4} \ln |x - 1| - \frac{1}{4} \ln |x + 1| - \frac{1}{2} \tan^{-1} x + C
\]