Solution for Midterm II [1]

(1) (a) $2(0+1+0+(-1))=0$
(b) $-1+2\cdot 1+2\cdot(-1)+2\cdot 0+1=0$
(c) $(-1+4\cdot 0+2\cdot 1+4\cdot 1+2\cdot(-1)+4\cdot 0+2\cdot 0+4\cdot(-1)+1)/3=0$

(2) (a) Convergent since $0\le \sin^2 x\le 1 \Rightarrow$

$$0\le \frac{\sin^2 x}{1+x^2}\le \frac{1}{1+x^2}$$

and

$$\int_1^\infty \frac{dx}{1+x^2}=\tan^{-1}x\Big]_1^\infty=\frac{\pi}{4}$$

is convergent.

(b) Convergent since $x+e^x>e^x$ for $x\ge 1\Rightarrow$

$$0\le \frac{1}{x+e^x}\le e^{-x}$$

and $\int_1^\infty e^{-x}dx=-e^{-x}]_1^\infty=e^{-1}$ is convergent.

(c) Divergent since $\sin x\le x$ for $0\le x\le 1\Rightarrow$

$$\frac{1}{x\sin x}\ge\frac{1}{x^2}$$

and

$$\int_0^1\frac{dx}{x^2}=\lim_{t\to 0^+}\int_t^1\frac{dx}{x^2}=\lim_{t\to 0^+}\left(\frac{1}{t}-1\right)=\infty$$

is divergent.

(3) (a) Use the method of cylindrical shell

$$2\pi\int_1^4 y\sqrt{y}dy=\frac{4\pi}{5}y^{5/2}\Big]_1^4=\frac{124\pi}{5}.$$

(b)

$$\pi\int_1^4(\sqrt{y})^2dy=\frac{\pi}{2}y^2\Big]_1^4=\frac{15\pi}{2}.$$

(c)

$$(1000)(10)\pi\int_1^4(\sqrt{y})^2(4-y)dy=10000\pi\left(2y^2-\frac{1}{3}y^3\right)\Big]_1^4=90000\pi \text{ J}$$

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[1] http://www.math.ucsb.edu/~xichen/math3b00w/mid2sol.pdf

(4) (a) Substitue $x = e^t$ and then integrate by parts

$$\begin{aligned}\int x(\ln x)^2 dx &= \int t^2 e^{2t} dt \\ &= \frac{1}{2}\int t^2 de^{2t} = \frac{1}{2}t^2 e^{2t} - \int te^{2t} dt \\ &= \frac{1}{2}t^2 e^{2t} - \frac{1}{2}(te^{2t} - \int e^{2t} dt) \\ &= \frac{1}{2}t^2 e^{2t} - \frac{1}{2}te^{2t} + \frac{1}{4}e^{2t} + C \\ &= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 + C\end{aligned}$$

(b) Integrate by parts

$$\begin{aligned}\int_0^{\pi/2} x \sin 2x dx &= -\frac{1}{2}\int_0^{\pi/2} x d(\cos 2x) \\ &= -\frac{1}{2} \left. x \cos 2x\right]_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2} \cos 2x dx \\ &= \frac{\pi}{4} + \frac{1}{4} \left. \sin 2x\right]_0^{\pi/2} = \frac{\pi}{4}\end{aligned}$$

(c) Substitute $u = \cos x$

$$\begin{aligned}\int \sin^3 x dx &= \int \sin^2 x(\sin x dx) = -\int \sin^2 x d\cos x \\ &= -\int (1-u^2) du = \frac{1}{3}u^3 - u + C \\ &= \frac{1}{3}\cos^3 x - \cos x + C\end{aligned}$$

(d) Substitute $u = e^t$ and then integrate a rational function by expressing it as a sum of partial fractions

$$\begin{aligned}\int \frac{1}{e^t+1} dt &= \int \frac{1}{e^t(e^t+1)} de^t \\ &= \int \frac{1}{u(u+1)} du = \int \frac{1}{u} du - \int \frac{1}{u+1} du \\ &= \ln|u| - \ln|u+1| + C = t - \ln(e^t+1) + C\end{aligned}$$

(e) Express the integrant as a sum of partial fractions

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+c}{x^2-x+1}.$$

Multiply x^3+1 on both sides and we have

$$\begin{aligned}1 &= A(x^2-x+1)+(Bx+C)(x+1)\\ &= (A+B)x^2+(B+C-A)x+(A+C)\end{aligned}$$

Hence $A+B=0$, $B+C-A=0$ and $A+C=1$. Solve the system of linear equations and we have $A=1/3$, $B=-1/3$ and $C=2/3$. So

$$\begin{aligned}\int \frac{1}{x^3+1}dx &= \frac{1}{3}\int \frac{1}{x+1}dx - \frac{1}{3}\int \frac{x-2}{x^2-x+1}dx\\ &= \frac{1}{3}\ln|x+1| - \frac{1}{3}\int \frac{(x-\frac{1}{2})-\frac{3}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}}dx\\ &= \frac{1}{3}\ln|x+1| - \frac{1}{3}\int \frac{x-\frac{1}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}}dx\\ &\quad + \frac{1}{2}\int \frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}}dx\\ &= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln\left(\left(x-\frac{1}{2}\right)^2+\frac{3}{4}\right)\\ &\quad + \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{\sqrt{3}}{3}(2x-1)\right)+C\\ &= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln(x^2-x+1)\\ &\quad + \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{\sqrt{3}}{3}(2x-1)\right)+C\end{aligned}$$

A Reminder for Time & Location of the Final: Mar. 20, 7:30 - 10:30pm, Phelps 1260