(1) (15 points) Find the tangent line of the curve \( x^3y + 2xy^3 = 3 \) at the point \((1,1)\).

(2) (10 points) Let \( f(x) = \sec x \). Find \( f'''(0) \).

(3) (30 points) Find the derivatives of the following functions.
   (a) (10 points) \( f(x) = \ln(\ln x) \).
   (b) (10 points) \( f(x) = \sqrt{\frac{1+x}{1-x}} \).
   (c) (10 points) \( f(x) = (\cos^{-1} x + \pi)^x \).

(4) (15 points) Let \( f(x) \) and \( g(x) \) be two functions that are twice-differentiable at 0 and \( F(x) = f(x)g(x) \). Suppose that \( f(0) = g(0) = 1 \), \( f'(0) = g'(0) = 2 \) and \( f''(0) = g''(0) = 3 \). Find \( F''(0) \).

(5) (10 points) Find the limit
\[
\lim_{x \to 0} x \cot x.
\]

(6) (20 points) Water is leaking out of an inverted conical tank at a rate of 10,000 \( \text{cm}^3/\text{min} \) at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of 20 cm/min when the height of the water is 2 m, find the rate at which water is being pumped into the tank.
Solutions

(1) (Implicit Differentiation) Take derivatives on both sides with respect to $x$:
\[
\frac{d}{dx}(x^3 y + 2xy^3) = \frac{d}{dx}(3)
\]
\[
\Rightarrow \frac{d}{dx}(x^3 y) + \frac{d}{dx}(2xy^3) = 0
\]
\[
\Rightarrow \left(\frac{d}{dx}x^3\right) y + x^3 \frac{dy}{dx} + \left(\frac{d}{dx}2x\right) y^3 + 2x\left(\frac{d}{dx}y^3\right) = 0
\]
\[
\Rightarrow 3x^2y + x^3 \frac{dy}{dx} + 2y^3 + 2x(3y^2) \frac{dy}{dx} = 0
\]
\[
\Rightarrow (x^3 + 6xy^2) \frac{dy}{dx} = -(3x^2y + 2y^3)
\]
\[
\Rightarrow \frac{dy}{dx} = -\frac{3x^2y + 2y^3}{x^3 + 6xy^2}.
\]
When $x = y = 1$, $dy/dx = -5/7$. So the tangent line is $5(x - 1) + 7(y - 1) = 0$.

(2) Calculate $f'''(x)$: $(\sec x)''' = (\sec x \tan x)'' = (\sec x \tan^2 x + \sec^3 x)' = \sec x \tan^3 x + 2\sec^3 x \tan x + 3\sec^3 x \tan x$. So $f'''(0) = 0$.

(3) (a) (Chain rule and derivatives of logarithmic functions) $(\ln(\ln x))' = (\ln x)^{-1}(\ln x)' = (x \ln x)^{-1}$.

(b) (Chain rule and quotient rule)
\[
\left(\sqrt{\frac{1+x}{1-x}}\right)' = \frac{1}{2} \sqrt{\frac{1-x}{1+x}} \left(\frac{1+x}{1-x}\right)' = \frac{1}{2} \sqrt{\frac{1-x}{1+x}} \cdot \frac{(1+x)'(1-x) - (1+x)(1-x)'}{(1-x)^2} = \frac{1}{2} \sqrt{\frac{1-x}{1+x}} \cdot \frac{2}{(1-x)^2} = (1+x)^{-1/2}(1-x)^{-3/2}.
\]
Probably it will be a little easier to use logarithmic differentiation here. Let
\[
y = \sqrt{\frac{1+x}{1-x}}.
\]
So
\[ \ln y = \frac{1}{2} \ln(1 + x) - \frac{1}{2} \ln(1 - x). \]

Take derivative on both sides with respect to \( x \):
\[ \frac{1}{y} \frac{dy}{dx} = \frac{1}{2(1 + x)} + \frac{1}{2(1 - x)}. \]

So
\[ \frac{dy}{dx} = y \left( \frac{1}{2(1 + x)} + \frac{1}{2(1 - x)} \right) \]
\[ = \sqrt{\frac{1 + x}{1 - x}} \cdot \frac{1}{2(1 + x) + 2(1 - x)} \]
\[ = \sqrt{\frac{1 + x}{1 - x}} \cdot \frac{1}{(1 - x)(1 + x)} \]
\[ = (1 + x)^{-1/2}(1 - x)^{-3/2}. \]

(c) (Logarithmic differentiation) Let \( y = (\cos^{-1} x + \pi)^x \). Then \( \ln y = x \ln(\cos^{-1} x + \pi) \). Take derivatives on both sides with respect to \( x \):
\[ \frac{1}{y} \frac{dy}{dx} = \ln(\cos^{-1} x + \pi) + x(\ln(\cos^{-1} x + \pi))' \]
\[ = \ln(\cos^{-1} x + \pi) + \frac{x}{\cos^{-1} x + \pi}(\cos^{-1} x + \pi)' \]
\[ = \ln(\cos^{-1} x + \pi) + \frac{x}{\cos^{-1} x + \pi} \left( -\frac{1}{\sqrt{1 - x^2}} \right) \]
\[ = \ln(\cos^{-1} x + \pi) - \frac{x}{\sqrt{1 - x^2} (\cos^{-1} x + \pi)}. \]

So
\[ \frac{dy}{dx} = (\cos^{-1} x + \pi)^x \left( \ln(\cos^{-1} x + \pi) - \frac{x}{\sqrt{1 - x^2} (\cos^{-1} x + \pi)} \right). \]

(4) By product rule, \( F'(x) = (f(x)g(x))' = f'(x)g(x) + f(x)g'(x) \)
and \( F''(x) = (f'(x)g(x))' + (f(x)g'(x))' = f''(x)g(x) + f'(x)g'(x) + f'(x)g'(x) + f(x)g''(x) = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x). \)
So \( F''(0) = 3 \cdot 1 + 2 \cdot 2 \cdot 2 + 1 \cdot 3 = 14. \)
(5) Use the fact $\lim_{x \to 0} \frac{\sin x}{x} = 1$:

$$\lim_{x \to 0} x \cot x = \lim_{x \to 0} x \left( \frac{\cos x}{\sin x} \right) = \lim_{x \to 0} \cos x \left( \lim_{x \to 0} \frac{x}{\sin x} \right) = (\lim_{x \to 0} \cos x) \left( \lim_{x \to 0} \frac{x}{\sin x} \right) = 1,$$

where

$$\lim_{x \to 0} \frac{x}{\sin x} = \left( \lim_{x \to 0} \frac{\sin x}{x} \right)^{-1} = 1.$$

(6) Let $h$ be the height of the water level, $r$ be the radius of the surface of the water and $V$ be the volume of the water at any moment. Then $r/h = 2/6$ and $V = \pi r^2 h/3$. So $r = h/3$ and $V = \pi h^3/27$. Therefore

$$\frac{dV}{dt} = \frac{\pi h^2}{9} \left( \frac{dh}{dt} \right).$$

When $h = 2$ m = 200 cm, $dh/dt = 20$ cm/min and $dV/dt = (\pi(200)^2/9) \cdot 20 = 800,000\pi/9$ cm$^3$/min. So water is pumped in the tank at a rate of $800,000\pi/9 + 10,000$ cm$^3$/min.