

- (1) (15 points) Find the tangent line of the curve $x^3y + 2xy^3 = 3$ at the point $(1, 1)$.
- (2) (10 points) Let $f(x) = \sec x$. Find $f'''(0)$.
- (3) (30 points) Find the derivatives of the following functions.
- (a) (10 points) $f(x) = \ln(\ln x)$.
- (b) (10 points) $f(x) = \sqrt{\frac{1+x}{1-x}}$.
- (c) (10 points) $f(x) = (\cos^{-1} x + \pi)^x$.
- (4) (15 points) Let $f(x)$ and $g(x)$ be two functions that are twice-differentiable at 0 and $F(x) = f(x)g(x)$. Suppose that $f(0) = g(0) = 1$, $f'(0) = g'(0) = 2$ and $f''(0) = g''(0) = 3$. Find $F''(0)$.
- (5) (10 points) Find the limit

$$\lim_{x \rightarrow 0} x \cot x.$$

- (6) (20 points) Water is leaking out of an inverted conical tank at a rate of $10,000 \text{ cm}^3/\text{min}$ at the same time that water is being pumped into the tank at a constant rate. The tank has height 6 m and the diameter at the top is 4 m. If the water level is rising at a rate of $20 \text{ cm}/\text{min}$ when the height of the water is 2 m, find the rate at which water is being pumped into the tank.

Solutions

- (1) (Implicit Differentiation) Take derivatives on both sides with respect to x :

$$\begin{aligned} \frac{d}{dx}(x^3y + 2xy^3) &= \frac{d}{dx}(3) \\ \Rightarrow \frac{d}{dx}(x^3y) + \frac{d}{dx}(2xy^3) &= 0 \\ \Rightarrow \left(\frac{d}{dx}x^3\right)y + x^3\frac{dy}{dx} + \left(\frac{d}{dx}2x\right)y^3 + 2x\left(\frac{d}{dx}y^3\right) &= 0 \\ \Rightarrow 3x^2y + x^3\frac{dy}{dx} + 2y^3 + 2x(3y^2)\frac{dy}{dx} &= 0 \\ \Rightarrow (x^3 + 6xy^2)\frac{dy}{dx} &= -(3x^2y + 2y^3) \\ \Rightarrow \frac{dy}{dx} &= -\frac{3x^2y + 2y^3}{x^3 + 6xy^2}. \end{aligned}$$

When $x = y = 1$, $dy/dx = -5/7$. So the tangent line is $5(x - 1) + 7(y - 1) = 0$.

- (2) Calculate $f'''(x)$: $(\sec x)''' = (\sec x \tan x)'' = (\sec x \tan^2 x + \sec^3 x)' = \sec x \tan^3 x + 2 \sec^3 x \tan x + 3 \sec^3 x \tan x$. So $f'''(0) = 0$.
- (3) (a) (Chain rule and derivatives of logarithmic functions) $(\ln(\ln x))' = (\ln x)^{-1}(\ln x)' = (x \ln x)^{-1}$.
- (b) (Chain rule and quotient rule)

$$\begin{aligned} \left(\sqrt{\frac{1+x}{1-x}}\right)' &= \frac{1}{2} \sqrt{\frac{1-x}{1+x}} \left(\frac{1+x}{1-x}\right)' \\ &= \frac{1}{2} \sqrt{\frac{1-x}{1+x}} \cdot \frac{(1+x)'(1-x) - (1+x)(1-x)'}{(1-x)^2} \\ &= \frac{1}{2} \sqrt{\frac{1-x}{1+x}} \cdot \frac{2}{(1-x)^2} \\ &= (1+x)^{-1/2}(1-x)^{-3/2}. \end{aligned}$$

Probably it will be a little easier to use logarithmic differentiation here. Let

$$y = \sqrt{\frac{1+x}{1-x}}.$$

So

$$\ln y = \frac{1}{2} \ln(1+x) - \frac{1}{2} \ln(1-x).$$

Take derivative on both sides with respect to x :

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{2(1+x)} + \frac{1}{2(1-x)}.$$

So

$$\begin{aligned} \frac{dy}{dx} &= y \left(\frac{1}{2(1+x)} + \frac{1}{2(1-x)} \right) \\ &= \sqrt{\frac{1+x}{1-x}} \cdot \left(\frac{1}{2(1+x)} + \frac{1}{2(1-x)} \right) \\ &= \sqrt{\frac{1+x}{1-x}} \cdot \frac{1}{(1-x)(1+x)} \\ &= (1+x)^{-1/2} (1-x)^{-3/2}. \end{aligned}$$

(c) (Logarithmic differentiation) Let $y = (\cos^{-1} x + \pi)^x$. Then $\ln y = x \ln(\cos^{-1} x + \pi)$. Take derivatives on both sides with respect to x :

$$\begin{aligned} \frac{1}{y} \frac{dy}{dx} &= \ln(\cos^{-1} x + \pi) + x(\ln(\cos^{-1} x + \pi))' \\ &= \ln(\cos^{-1} x + \pi) + \frac{x}{\cos^{-1} x + \pi} (\cos^{-1} x + \pi)' \\ &= \ln(\cos^{-1} x + \pi) + \frac{x}{\cos^{-1} x + \pi} \left(-\frac{1}{\sqrt{1-x^2}} \right) \\ &= \ln(\cos^{-1} x + \pi) - \frac{x}{\sqrt{1-x^2}(\cos^{-1} x + \pi)}. \end{aligned}$$

So

$$\frac{dy}{dx} = (\cos^{-1} x + \pi)^x \left(\ln(\cos^{-1} x + \pi) - \frac{x}{\sqrt{1-x^2}(\cos^{-1} x + \pi)} \right).$$

- (4) By product rule, $F'(x) = (f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$
 and $F''(x) = (f'(x)g(x))' + (f(x)g'(x))' = f''(x)g(x) + f'(x)g'(x) + f'(x)g'(x) + f(x)g''(x) = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)$.
 So $F''(0) = 3 \cdot 1 + 2 \cdot 2 \cdot 2 + 1 \cdot 3 = 14$.

(5) Use the fact $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$:

$$\begin{aligned} \lim_{x \rightarrow 0} x \cot x &= \lim_{x \rightarrow 0} x \left(\frac{\cos x}{\sin x} \right) \\ &= \lim_{x \rightarrow 0} \cos x \left(\frac{x}{\sin x} \right) \\ &= \left(\lim_{x \rightarrow 0} \cos x \right) \left(\lim_{x \rightarrow 0} \frac{x}{\sin x} \right) = 1, \end{aligned}$$

where

$$\lim_{x \rightarrow 0} \frac{x}{\sin x} = \left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right)^{-1} = 1.$$

(6) Let h be the height of the water level, r be the radius of the surface of the water and V be the volume of the water at any moment. Then $r/h = 2/6$ and $V = \pi r^2 h/3$. So $r = h/3$ and $V = \pi h^3/27$. Therefore

$$\frac{dV}{dt} = \frac{\pi h^2}{9} \left(\frac{dh}{dt} \right).$$

When $h = 2 \text{ m} = 200 \text{ cm}$, $dh/dt = 20 \text{ cm/min}$ and $dV/dt = (\pi(200)^2/9) \cdot 20 = 800,000\pi/9 \text{ cm}^3/\text{min}$. So water is pumped in the tank at a rate of $800,000\pi/9 + 10,000 \text{ cm}^3/\text{min}$.