Solution for Practice Final

(1) The graph of \( y = 3^x - 1 + 4 \) is the graph of \( y = 3^x \) shifted up by 4 and right by 1. Reflect it by the line \( y = x \) and we obtain the graph of its inverse.

(2) (Implicit Differentiation) Take derivatives on both sides with respect to \( x \):

\[
\frac{d}{dx} (x^3 y + 2xy^3) = \frac{d}{dx} (3) \\
\Rightarrow \frac{d}{dx} (x^3 y) + \frac{d}{dx} (2xy^3) = 0 \\
\Rightarrow \left( \frac{d}{dx} x^3 \right) y + x^3 \frac{dy}{dx} + \left( \frac{d}{dx} 2x \right) y^3 + 2x \left( \frac{d}{dx} y^3 \right) = 0 \\
\Rightarrow 3x^2 y + x^3 \frac{dy}{dx} + 2y^3 + 2x(3y^2) \frac{dy}{dx} = 0 \\
\Rightarrow (x^3 + 6xy^2) \frac{dy}{dx} = -(3x^2 y + 2y^3) \\
\Rightarrow \frac{dy}{dx} = -\frac{3x^2 y + 2y^3}{x^3 + 6xy^2}.
\]

When \( x = y = 1 \), \( dy/dx = -5/7 \). So the tangent line is \( 5(x - 1) + 7(y - 1) = 0 \).

(3) Here is a word of warning. You are asked to calculate these limits without using L’Hôpital’s rule. Not credit will be given if you are using L’Hôpital’s rule.

(i) \[
\lim_{x \to 5} \frac{\sqrt{3x + 1} - 4}{x - 5} = \lim_{x \to 5} \frac{(\sqrt{3x + 1} - 4)(\sqrt{3x + 1} + 4)}{(x - 5)(\sqrt{3x + 1} + 4)} \\
= \lim_{x \to 5} \frac{3(x - 5)}{(x - 5)(\sqrt{3x + 1} + 4)} \\
= \lim_{x \to 5} \frac{3}{\sqrt{3x + 1} + 4} = \frac{3}{8}.
\]
(ii)

\[
\lim_{x \to -\infty} \frac{\sqrt{4x^2 + 3x + 1}}{x + 2} = \lim_{x \to -\infty} \frac{\sqrt{4x^2 + 3 + 2/x}}{(x + 2)/x}
\]
\[
= \lim_{x \to -\infty} \frac{-\sqrt{(4x^2 + 3 + 2)/x^2}}{1 + (2/x)}
\]
\[
= -\frac{\lim_{x \to -\infty} \sqrt{4 + (3/x) + (2/x^2)}}{\lim_{x \to -\infty} (1 + (2/x))} = -2.
\]

(iii)

\[
\lim_{x \to 0} \frac{\sin 2x}{5x} = \frac{2}{5} \lim_{x \to 0} \frac{\sin 2x}{2x} = \frac{2}{5}.
\]

(4) By L'Hôpital's rule,

\[
\lim_{x \to \infty} \frac{\ln \ln x}{x} = \lim_{x \to \infty} \frac{(\ln \ln x)'}{x} = \lim_{x \to \infty} \frac{1}{x \ln x} = 0.
\]

(5) (i)

\[
(tan(3x^2) \sin x)' = (\tan(3x^2))' \sin x + tan(3x^2)(\sin x)'
\]
\[
= 6x \sec^2(3x^2) \sin x + tan(3x^2) \cos x.
\]

(ii)

\[
\left(\sqrt{\frac{\ln x + 1}{\ln x - 1}}\right)' = \frac{1}{2} \left(\sqrt{\frac{\ln x - 1}{\ln x + 1}}\right) \left(\frac{\ln x + 1}{\ln x - 1}\right)'
\]
\[
= \frac{1}{2} \left(\sqrt{\frac{\ln x - 1}{\ln x + 1}}\right) \frac{(\ln x + 1)'(\ln x - 1) - (\ln x + 1)(\ln x - 1)'}{(\ln x - 1)^2}
\]
\[
= \frac{1}{2} \left(\sqrt{\frac{\ln x - 1}{\ln x + 1}}\right) \frac{-2}{x(\ln x - 1)^2}
\]
\[
= -\frac{1}{x(\ln x + 1)^{1/2}(\ln x - 1)^{3/2}}.
\]
(iii) 
\[
\left( \frac{x^2 + e^{x^2}}{\tan x + \cos x} \right)'
\]
\[
= \frac{(x^2 + e^{x^2})'(\tan x + \cos x) - (x^2 + e^{x^2})(\tan x + \cos x)'}{(\tan x + \cos x)^2}
\]
\[
= \frac{(2x + 2xe^{x^2})(\tan x + \cos x) - (x^2 + e^{x^2})(\sec^2 x - \sin x)}{(\tan x + \cos x)^2}.
\]

(iv) Let \( y = (\cos^{-1} x)^x \). Then \( \ln y = x \ln(\cos^{-1} x) \). Take derivatives on both sides with respect to \( x \):

\[
\frac{1}{y} \frac{dy}{dx} = \ln(\cos^{-1} x) + x(\ln(\cos^{-1} x))'
\]
\[
= \ln(\cos^{-1} x) + \frac{x}{\cos^{-1} x} (\cos^{-1} x)'
\]
\[
= \ln(\cos^{-1} x) + \frac{x}{\cos^{-1} x} \left( - \frac{1}{\sqrt{1 - x^2}} \right)
\]
\[
= \ln(\cos^{-1} x) - \frac{x}{\sqrt{1 - x^2}(\cos^{-1} x)}.
\]

So
\[
\frac{dy}{dx} = (\cos^{-1} x)^x \left( \ln(\cos^{-1} x) - \frac{x}{\sqrt{1 - x^2}(\cos^{-1} x)} \right).
\]

(6) Since
\[
f'(x) = 4x^3 + 12x^2 + 4x
\]
\[
= 4x(x^2 + 3x + 1)
\]
\[
= 4x \left( x - \frac{-3 + \sqrt{5}}{2} \right) \left( x - \frac{-3 - \sqrt{5}}{2} \right)
\]
f(\(x\)) has three critical numbers 0, \((-3 + \sqrt{5})/2\) and \((-3 - \sqrt{5})/2\).

\(f'(x) < 0\) for \( x < (-3 - \sqrt{5})/2\), \(f'(x) > 0\) for \((-3 - \sqrt{5})/2 < x < (-3 + \sqrt{5})/2\), \(f'(x) < 0\) for \((-3 + \sqrt{5})/2 < x < 0\) and \(f'(x) > 0\) for \(x > 0\). So by the first derivative test, \(f(x)\) has local minima at \((-3 - \sqrt{5})/2\) and 0 and \(f(x)\) has a local maximum at \((-3 + \sqrt{5})/2\).

Alternative, you may use the second derivative test: \(f''(x) = 12x^2 + 24x + 4, f''(0) = 4 > 0\), \(f''((-3 + \sqrt{5})/2) = -8 - 6\sqrt{5} < 0\) and \(f''((-3 - \sqrt{5})/2) = -8 + 6\sqrt{5} > 0\). So \(f(x)\) has local
minima at \((-3 - \sqrt{5})/2\) and 0 and \(f(x)\) has a local maximum at \((-3 + \sqrt{5})/2\).

(7) (a) The domain of \(f(x) = (x^2 + 1)/(x^2 - 1)\) is \((-\infty, -1) \cup (-1, 1) \cup (1, \infty)\).

The curve \(y = f(x)\) has no \(x\)-intercepts and its \(y\)-intercept is \((0, -1)\). Since \(f(-x) = f(x)\), \(f(x)\) is even and \(f(x)\) is not odd. Obviously, \(f(x)\) is not periodic.

There are two vertical asymptotes \(x = -1\) and \(x = 1\) since

\[
\lim_{x \to (-1)^-} f(x) = \lim_{x \to -1^+} f(x) = \infty
\]

and

\[
\lim_{x \to (-1)^+} f(x) = \lim_{x \to -1^-} f(x) = -\infty.
\]

Since

\[
\lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{x^2 + 1}{x(x^2 - 1)} = \lim_{x \to \infty} \frac{(x^2 + 1)/x^3}{(x^2 - 1)/x^2} = \lim_{x \to \infty} \frac{1/x + 1/x^3}{1 - 1/x^2} = 0
\]

and

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^2 + 1}{x^2 - 1} = \lim_{x \to \infty} \frac{(x^2 + 1)/x^2}{(x^2 - 1)/x^2} = \lim_{x \to \infty} \frac{1 + 1/x^2}{1 - 1/x^2} = 1,
\]

and similarly, \(\lim_{x \to -\infty} f(x)/x = 1\) and \(\lim_{x \to -\infty} f(x) = 0\), it has a horizontal asymptote \(y = 1\).

Since

\[
\left(\frac{x^2 + 1}{x^2 - 1}\right)' = \left(1 + \frac{2}{x^2 - 1}\right)' = -\frac{4x}{(x^2 - 1)^2},
\]

\(f'(x) < 0\) for \(x > 0\) and \(f'(x) > 0\) for \(x < 0\). So \(f(x)\) is increasing on \((-\infty, -1)\) and \((-1, 0)\) and decreasing on \((0, 1)\) and \((1, \infty)\).
By the first derivative test, \( f(x) \) has a local maximum at \( x = 0 \).

Since

\[
\left( \frac{x^2 + 1}{x^2 - 1} \right)'' = - \left( \frac{4x}{(x^2 - 1)^2} \right)',
\]

\[
= \frac{12x^2 + 4}{(x^2 - 1)^3},
\]

\( f''(x) > 0 \) for \( x > 1 \) or \( x < -1 \) and \( f''(x) < 0 \) for \( -1 < x < 1 \). So \( f(x) \) is concave upward on \((1, \infty)\) and \((-\infty, -1)\) and concave downward on \((-1, 1)\). It has no point of inflection (\( f(x) \) is not defined at \( x = -1 \) and \( x = 1 \)).

(b) The domain of \( \ln x/x \) is \((0, \infty)\).

The \( x \)-intercept is \((1, 0)\) and it has no \( y \)-intercept (\( f(x) \) is not defined at \( x = 0 \)).

It is not even, not odd and not periodic. Actually, \( f(x) \) is only defined on \((0, \infty)\). It does not make much sense to talk about whether \( f(x) \) has these properties.

It has a vertical asymptote \( x = 0 \) since \( \lim_{x \to 0^+} f(x) = -\infty \).

Since

\[
\lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{\ln x}{x^2}
\]

\[
= \lim_{x \to \infty} \frac{(\ln x)'}{(x^2)'}
\]

\[
= \lim_{x \to \infty} \frac{1}{2x^2} = 0
\]

and

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{\ln x}{x}
\]

\[
= \lim_{x \to \infty} \frac{(\ln x)'}{(x)'}
\]

\[
= \lim_{x \to \infty} \frac{1}{x} = 0
\]

the curve has a horizontal asymptote \( y = 0 \).

Since

\[
\left( \frac{\ln x}{x} \right)' = \frac{(\ln x)\cdot x - (\ln x)(x)'}{x^2}
\]

\[
= \frac{1 - \ln x}{x^2}
\]
\( f'(x) > 0 \) for \( x < e \) and \( f'(x) < 0 \) for \( x > e \). So \( f(x) \) is increasing on \((0, e)\) and decreasing on \((e, \infty)\).

By the first derivative test, \( f(x) \) has a local maximum at \( x = e \).

Since
\[
\left( \frac{\ln x}{x} \right)'' = \left( \frac{1 - \ln x}{x^2} \right)'
= \frac{(1 - \ln x)x^2 - (1 - \ln x)(x^2)'}{x^4}
= \frac{2 \ln x - 3}{x^3}
\]
\( f''(x) > 0 \) for \( x > e^{3/2} \) and \( f''(x) < 0 \) for \( x < e^{3/2} \). So \( f(x) \) is concave upward on \((e^{3/2}, \infty)\) and concave downward on \((0, e^{3/2})\). It has a point of inflection at \( x = e^{3/2} \).

(8) Let \( h \) be the height of the water level, \( r \) be the radius of the surface of the water and \( V \) be the volume of the water at any moment. Then \( r/h = 2/6 \) and \( V = \pi r^2 h/3 \). So \( r = h/3 \) and \( V = \pi h^3/27 \). Therefore
\[
\frac{dV}{dt} = \frac{\pi h^2}{9} \left( \frac{dh}{dt} \right).
\]
When \( h = 2 \) m = 200 cm, \( dh/dt = 20 \) cm/min and \( dV/dt = (\pi (200)^2 / 9) \cdot 20 = 800,000 \pi / 9 \) cm³/min. So water is pumped in the tank at a rate of 800,000 \( \pi / 9 + 10,000 \) cm³/min.

(9) Calculate \( f'''(x) \):
\[
(\sec x)''' = (\sec x \tan x)''' = (\sec x \tan^2 x + \sec^3 x)'
= \sec x \tan^3 x + 2 \sec^3 x \tan x + 3 \sec^3 x \tan x
= \sec x \tan^3 x + 5 \sec^3 x \tan x.
\]
So \( f'''(\pi/4) = 11\sqrt{2} \).

(10) Let \( x \) be the distance between the bottom of the ladder and the wall. Then \( \sin \theta = x/10 \). Differentiate both sides of \( \sin \theta = x/10 \) with respect to \( t \):
\[
\cos \theta \frac{d\theta}{dt} = \frac{1}{10} \frac{dx}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{1}{10 \cos \theta} \frac{dx}{dt}.
\]
When \( \theta = \pi/3 \), \( \cos \theta = 1/2 \), \( dx/dt = 1 \) and \( d\theta/dt = 1/5 \) rad/s.
(11) Let $x$ be the length of one side of a rectangle inscribed inside a circle of radius $r$. Then the other side of the rectangle has length $\sqrt{4r^2 - x^2}$ and the area of the rectangle $A(x) = x\sqrt{4r^2 - x^2}$. We want to maximize $A(x)$ for $0 \leq x \leq 2r$. Since

$$A'(x) = \sqrt{4r^2 - x^2} - \frac{x^2}{\sqrt{4r^2 - x^2}} = \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}},$$

$A(x)$ has a critical point at $x = \sqrt{2}r$. Since $A(0) = A(2r) = 0$ and $A(\sqrt{2}r) = 2r^2$, $A(x)$ achieves maximum $2r^2$ at $x = \sqrt{2}r$. So among the rectangle inscribed inside a circle of radius, the rectangle with two sides $\sqrt{2}r$ has the largest area.

(12) Since $f(0) = -1$ and $f(2) = 9$, $(f(2) - f(0))/(2 - 0) = 5$. We want to find a number $c$ in $(0, 2)$ such that $f'(c) = 5$, i.e., $3c^2 + 1 = 5$. So $c = 2\sqrt{3}/3$.

(13) Let $\theta$ be the angle between the ladder and the ground. Then the length of the ladder is

$$f(\theta) = \frac{8}{\sin \theta} + \frac{4}{\cos \theta}.$$

We want to minimize $f(\theta)$ for $0 < \theta < \pi/2$. Since

$$f'(\theta) = -\frac{8 \cos \theta}{\sin^2 \theta} + \frac{4 \sin \theta}{\cos^2 \theta},$$

$f(\theta)$ has a critical point at $8 \cos \theta/\sin^2 \theta = 4 \sin \theta/\cos^2 \theta$, i.e., $2 \cos^3 \theta = \sin^3 \theta \Rightarrow \tan \theta = \sqrt{2}$. When $\tan \theta = \sqrt{2}$,

$$\sin \theta = \frac{\sqrt{2}}{\sqrt{3/4 + 1}} \text{ and } \cos \theta = \frac{1}{\sqrt{3/4 + 1}},$$

and hence $f(\theta) = 4(4^{1/3} + 1)^{3/2}$. Since

$$\lim_{\theta \to 0^+} f(\theta) = \lim_{\theta \to -\pi/2^-} f(\theta) = \infty,$$

$f(\theta)$ achieves the minimum $4(4^{1/3} + 1)^{3/2}$ at $\theta = \tan^{-1}(\sqrt{2})$. 