Solution for Midterm II (BRDA)

(1) Differentiate both sides of $2\cos x \sin y = 1$ with respect to x:

$$\begin{aligned}
&\frac{d}{dx}(2\cos x \sin y) = 0 \\
&\Rightarrow \left(\frac{d}{dx}\cos x\right)\sin y + \cos x\left(\frac{d}{dx}\sin y\right) = 0 \\
&\Rightarrow -\sin x \sin y + \cos x \cos y \frac{dy}{dx} = 0 \\
&\Rightarrow \frac{dy}{dx} = \frac{\sin x \sin y}{\cos x \cos y}.
\end{aligned}$$

When $x = y = \pi/4$, $dy/dx = 1$. So the corresponding tangent line is $y - \pi/4 = x - \pi/4$, i.e., $y = x$.

(2) $f'(x) = \sec^2 x$

$$f''(x) = 2\sec x(\sec x)' = 2\sec x(\sec x \tan x) = 2\sec^2 x \tan x$$

and

$$\begin{aligned}
f'''(x) &= 4\sec x(\sec x \tan x)\tan x + 2\sec^2 x(\sec^2 x) \\
&= 4\sec^2 x \tan^2 x + 2\sec^4 x.
\end{aligned}$$

So $f'''(\pi/4) = 4(\sqrt{2})^2 + 2(\sqrt{2})^4 = 16$.

(3) (a) $\left(\frac{\sin x}{x}\right)' = \frac{(\sin x)'x - \sin x(x)'}{x^2} = \frac{x\cos x - \sin x}{x^2}$.

(b)

$$\begin{aligned}
&\left(\sqrt{x + \sqrt{x + \sqrt{x}}}\right)' \\
&= \frac{1}{2}\left(x + \sqrt{x + \sqrt{x}}\right)^{-1/2}\left(x + \sqrt{x + \sqrt{x}}\right)' \\
&= \frac{1}{2}\left(x + \sqrt{x + \sqrt{x}}\right)^{-1/2}\left(1 + \frac{1}{2}(x + \sqrt{x})^{-1/2}\left(x + \sqrt{x}\right)'\right) \\
&= \frac{1}{2}\left(x + \sqrt{x + \sqrt{x}}\right)^{-1/2}\left(1 + \frac{1}{2}(x + \sqrt{x})^{-1/2}\left(1 + \frac{1}{2}x^{-1/2}\right)\right).
\end{aligned}$$

(c) Let $y = (\cos x)^x$. Then $\ln y = x \ln(\cos x)$. Differentiate both sides of $\ln y = x \ln(\cos x)$ with respect to x:

$$\frac{1}{y} \cdot \frac{dy}{dx} = \ln(\cos x) + x \frac{-\sin x}{\cos x}$$
$$\Rightarrow \frac{dy}{dx} = y\left(\ln(\cos x) - x \tan x\right).$$

So $((\cos x)^x)' = (\cos x)^x (\ln(\cos x) - x \tan x)$.

(4) $F'(x) = -[f(x)]^{-2} f'(x)$ and

$$\begin{aligned} F''(x) &= -\left([f(x)]^{-2}\right)' f'(x) - [f(x)]^{-2} f''(x) \\ &= 2[f(x)]^{-3}[f'(x)]^2 - [f(x)]^{-2} f''(x). \end{aligned}$$

So $F''(0) = 2 \cdot 1^{-3} \cdot 2^2 - 1^{-2} \cdot 3 = 5$.

(5) Let x be the distance traveled by ship A and y be the distance traveled by ship B after t hours. Let z be the distance between the two ships after t hours. Then $dx/dt = 35$, $dy/dt = 25$ and

$$(150 - x)^2 + y^2 = z^2.$$

Differentiate $(150 - x)^2 + y^2 = z^2$ on both sides with respect to t:

$$-2(150 - x)\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}.$$

When $t = 4$, $x = 35 \cdot 4 = 140$, $y = 25 \cdot 4 = 100$ and $z = \sqrt{(150 - 140)^2 + 100^2} = 10\sqrt{101}$. So

$$\begin{aligned} \frac{dz}{dt} &= -\left(\frac{150 - x}{z}\right)\frac{dx}{dt} + \left(\frac{y}{z}\right)\frac{dy}{dt} \\ &= -\frac{150 - 140}{10\sqrt{101}} \cdot 35 + \frac{100}{10\sqrt{101}} \cdot 25 = \frac{215}{\sqrt{101}} \end{aligned}$$

when $t = 4$. So the distance is changing at a rate of $215/\sqrt{101}$ km/h, which is about 21.39 km/h, at 4 P.M.