Solution for Midterm I (BRDA)

(1) $\ln(\ln x) = 1 \Rightarrow e^{\ln(\ln x)} = e \Rightarrow \ln x = e \Rightarrow e^{\ln x} = e^e \Rightarrow x = e^e$.

(2) Solve $y = \dfrac{x+1}{x-1}$ for x: $y(x-1) = x+1 \Rightarrow yx - x = y+1 \Rightarrow (y-1)x = y+1 \Rightarrow x = \dfrac{y+1}{y-1}$. So $f^{-1}(x) = \dfrac{x+1}{x-1}$.

The domains of $f(x)$ and $f^{-1}(x)$ (they are actually the same function) are both $\{x \neq 1\} = (-\infty, 1) \cup (1, \infty)$. So their ranges are both $(-\infty, 1) \cup (1, \infty)$.

Take the limits of $f(x)$ and $f^{-1}(x)$ as $x \to \infty$ and $x \to -\infty$:

$$\lim_{x\to\infty} \frac{x+1}{x-1} = \lim_{x\to\infty} \frac{(x+1)/x}{(x-1)/x} = \lim_{x\to\infty} \frac{1+\frac{1}{x}}{1-\frac{1}{x}} = 1,$$

$$\lim_{x\to-\infty} \frac{x+1}{x-1} = \lim_{x\to-\infty} \frac{(x+1)/x}{(x-1)/x} = \lim_{x\to-\infty} \frac{1+\frac{1}{x}}{1-\frac{1}{x}} = 1,$$

So there is only one horizontal asymptote $y = 1$.

(3) The slope of the tangent line of $y = x^3$ at $(1, 1)$ is:

$$\lim_{h\to 0} \frac{(1+h)^3 - 1}{h} = \lim_{h\to 0} \frac{3h + 3h^2 + h^3}{h} = \lim_{h\to 0}(3 + 3h + h^2) = 3.$$

So the tangent line of $y = x^3$ at $(1, 1)$ is $y - 1 = 3(x-1)$.

(4) Since

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sin x = 0$$

and

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \cos x = 1,$$

$\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$ and hence $\lim_{x\to 0} f(x)$ does not exist. So $f(x)$ is not continuous at $x = 0$ and $f(x)$ is not continuous on $(-\infty, \infty)$.

(5) Since

$$
\begin{aligned}
&\lim_{x\to\infty}(\sqrt{x^2+x+2}-\sqrt{x^2+1}) \\
&= \lim_{x\to\infty}\frac{(\sqrt{x^2+x+2}-\sqrt{x^2+1})(\sqrt{x^2+x+2}+\sqrt{x^2+1})}{\sqrt{x^2+x+2}+\sqrt{x^2+1}} \\
&= \lim_{x\to\infty}\frac{x+1}{\sqrt{x^2+x+2}+\sqrt{x^2+1}} \\
&= \lim_{x\to\infty}\frac{(x+1)/x}{(\sqrt{x^2+x+2}+\sqrt{x^2+1})/x} \\
&= \lim_{x\to\infty}\frac{1+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{2}{x^2}}+\sqrt{1+\frac{1}{x^2}}} = \frac{1}{2}
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{x\to-\infty}(\sqrt{x^2+x+2}-\sqrt{x^2+1}) \\
&= \lim_{x\to-\infty}\frac{(\sqrt{x^2+x+2}-\sqrt{x^2+1})(\sqrt{x^2+x+2}+\sqrt{x^2+1})}{\sqrt{x^2+x+2}+\sqrt{x^2+1}} \\
&= \lim_{x\to-\infty}\frac{x+1}{\sqrt{x^2+x+2}+\sqrt{x^2+1}} \\
&= \lim_{x\to-\infty}\frac{(x+1)/x}{(\sqrt{x^2+x+2}+\sqrt{x^2+1})/x} \\
&= \lim_{x\to-\infty}-\frac{1+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{2}{x^2}}+\sqrt{1+\frac{1}{x^2}}} = -\frac{1}{2}
\end{aligned}
$$

the horizontal asymptotes are $y = 1/2$ and $y = -1/2$. Notice that $\sqrt{x^2} = |x| = x$ if $x \geq 0$ and $-x$ if $x < 0$.

(6)

$$
\begin{aligned}
\lim_{x\to 1}\frac{x^2-4x+3}{x^2-3x+2} &= \lim_{x\to 1}\frac{(x-1)(x-3)}{(x-1)(x-2)} \\
&= \lim_{x\to 1}\frac{x-3}{x-2} = 2.
\end{aligned}
$$

$$
\lim_{x\to 0^-} e^{1/x} = \lim_{t\to-\infty} e^t = 0.
$$