Solution for Midterm I (NH)

(1) $\ln x + \ln(x-2) = 1 \Rightarrow \ln(x(x-2)) = 1 \Rightarrow e^{\ln(x(x-2))} = e \Rightarrow x(x-2) = e$. Solve the quadratic equation $x^2 - 2x - e = 0$ and we obtain $x = 1 \pm \sqrt{1+e}$. Since $x > 2$ and $1 - \sqrt{1+e} < 0$, $x = 1 + \sqrt{1+e}$.

(2) Solve $y = \dfrac{x-1}{x+1}$ for x: $y(x+1) = x - 1 \Rightarrow yx - x = -1 - y \Rightarrow (y-1)x = -(y+1) \Rightarrow x = -\dfrac{y+1}{y-1}$. So $f^{-1}(x) = -\dfrac{x+1}{x-1}$.

The domains of $f(x)$ is $\{x \neq -1\} = (-\infty, -1) \cup (-1, \infty)$ and the domain of $f^{-1}(x)$ is $\{x \neq 1\} = (-\infty, 1) \cup (1, \infty)$. So the range of $f(x)$ is $(-\infty, 1) \cup (1, \infty)$ and the range of $f^{-1}(x)$ is $(-\infty, -1) \cup (-1, \infty)$.

Take the limits of $f(x)$ and $f^{-1}(x)$ as $x \to \infty$ and $x \to -\infty$:

$$\lim_{x\to\infty} \frac{x-1}{x+1} = \lim_{x\to\infty} \frac{(x-1)/x}{(x+1)/x} = \lim_{x\to\infty} \frac{1-\frac{1}{x}}{1+\frac{1}{x}} = 1,$$

$$\lim_{x\to-\infty} \frac{x-1}{x+1} = \lim_{x\to-\infty} \frac{(x-1)/x}{(x+1)/x} = \lim_{x\to-\infty} \frac{1-\frac{1}{x}}{1+\frac{1}{x}} = 1,$$

$$\lim_{x\to\infty} -\frac{x+1}{x-1} = -\lim_{x\to\infty} \frac{(x+1)/x}{(x-1)/x} = -\lim_{x\to\infty} \frac{1+\frac{1}{x}}{1-\frac{1}{x}} = -1,$$

$$\lim_{x\to-\infty} -\frac{x+1}{x-1} = -\lim_{x\to-\infty} \frac{(x+1)/x}{(x-1)/x} = -\lim_{x\to-\infty} \frac{1+\frac{1}{x}}{1-\frac{1}{x}} = -1,$$

So the only horizontal asymptote for $y = f(x)$ is $y = 1$ and the only horizontal asymptote for $y = f^{-1}(x)$ is $y = -1$.

(3) The slope of the tangent line of $y = 1/x$ at $(1,1)$ is:

$$\lim_{h\to0} \frac{\frac{1}{1+h} - 1}{h} = \lim_{h\to0} \frac{\frac{-h}{1+h}}{h} = \lim_{h\to0} -\frac{1}{1+h} = -1.$$

So the tangent line of $y = 1/x$ at $(1,1)$ is $y - 1 = -(x-1)$.

(4) Since

$$\lim_{x\to(\pi/4)^+} f(x) = \lim_{x\to(\pi/4)^+} \sin x = \sin\frac{\pi}{4}$$

and

$$\lim_{x\to(\pi/4)^-} f(x) = \lim_{x\to(\pi/4)^-} \cos x = \cos\frac{\pi}{4},$$

$\lim_{x\to(\pi/4)^+} f(x) = \lim_{x\to(\pi/4)^-} f(x) = \sin(\pi/4) = \cos(\pi/4) = f(\pi/4)$. So $f(x)$ is continuous at $x = \pi/4$ and $f(x)$ is obviously continuous on $\{x \neq \pi/4\}$. Therefore $f(x)$ is continuous on $(-\infty, \infty)$.

(5) Since

$$\begin{aligned}
&\lim_{x\to\infty}(\sqrt{x^2+x+2}-x) \\
&= \lim_{x\to\infty} \frac{(\sqrt{x^2+x+2}-x)(\sqrt{x^2+x+2}+x)}{\sqrt{x^2+x+2}+x} \\
&= \lim_{x\to\infty} \frac{x+2}{\sqrt{x^2+x+2}+x} \\
&= \lim_{x\to\infty} \frac{(x+2)/x}{(\sqrt{x^2+x+2}+x)/x} \\
&= \lim_{x\to\infty} \frac{1+\frac{2}{x}}{\sqrt{1+\frac{1}{x}+\frac{2}{x^2}}+1} = \frac{1}{2}
\end{aligned}$$

and

$$\lim_{x\to-\infty}(\sqrt{x^2+x+2}-x) = \infty - (-\infty) = \infty,$$

the only horizontal asymptote is $y = 1/2$.

(6)

$$\begin{aligned}
\lim_{x\to1} \frac{x^2-3x+2}{x^2-4x+3} &= \lim_{x\to1} \frac{(x-1)(x-2)}{(x-1)(x-3)} \\
&= \lim_{x\to1} \frac{x-2}{x-3} = \frac{1}{2}.
\end{aligned}$$

$$\lim_{x\to0^+} e^{-1/x} = \lim_{t\to\infty} e^{-t} = 0.$$