Solution for Midterm I [1]

1. (a) $2^x \cdot 2^{x+2} = 1 \Rightarrow 2^{2x+2} = 1 \Rightarrow 2x + 2 = 0 \Rightarrow x = -1$

   (b) $2^x + 2^{x+2} = 1 \Rightarrow 2^x + 2^x \cdot 2^2 = 1 \Rightarrow 2^x + 4(2^x) = 1 \Rightarrow 5(2^x) = 1 \Rightarrow 2^x = 1/5 \Rightarrow x = \log_2(1/5) = -(\ln 5)/(\ln 2)$.

2. (a) Solve $y = (e^x + 1)/(e^x - 1)$ for x:

$$\begin{aligned}
&\Rightarrow e^x + 1 = y(e^x - 1) \Rightarrow e^x + 1 = ye^x - y \\
&\Rightarrow y + 1 = ye^x - e^x \Rightarrow y + 1 = e^x(y - 1) \\
&\Rightarrow e^x = \frac{y+1}{y-1} \Rightarrow x = \ln\left(\frac{y+1}{y-1}\right).
\end{aligned}$$

   So the inverse function is

$$f^{-1}(x) = \ln\left(\frac{x+1}{x-1}\right).$$

3. (b) The domain of $f(x)$ is $\{e^x - 1 \neq 0\} = \{e^x \neq 1\} = \{x \neq 0\} = (-\infty, 0) \cup (0, \infty)$.

   The domain of $f^{-1}(x)$ is $\{(x+1)/(x-1) > 0\} = \{x > 1 \text{ or } x < -1\} = (-\infty, -1) \cup (1, \infty)$.

   The range of $f(x)$ is the same as the domain of $f^{-1}(x)$ and the range of $f^{-1}(x)$ is the same as the domain of $f(x)$.

	Domain	Range
$f(x)$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, -1) \cup (1, \infty)$
$f^{-1}(x)$	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, 0) \cup (0, \infty)$

4. The slope of the tangent line is

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1}(x + 1) = 2.$$

   So the tangent line is $y - 1 = 2(x - 1)$.

[1] http://www.math.ucsb.edu/~xichen/math3a01f/mid1s.pdf

5. Since $\lim_{x\to 1^+} f(x) = 1 - c$ and $\lim_{x\to 1^-} f(x) = c - 3$, $1 - c = c - 3$ if $f(x)$ is continuous. So $c = 2$.

6. Since

$$\lim_{x\to -2^-} \frac{x}{\sqrt{x^2+3x+2}} = -\infty \text{ and } \lim_{x\to -1^+} \frac{x}{\sqrt{x^2+3x+2}} = -\infty$$

the curve has two vertical asymptotes $x = -1$ and $x = -2$.

Since

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+3x+2}} = \lim_{x\to\infty} \frac{1}{\sqrt{x^2+3x+2}/x} = \lim_{x\to\infty} \frac{1}{\sqrt{(x^2+3x+2)/x^2}}$$
$$= \lim_{x\to\infty} \frac{1}{\sqrt{1+3/x+2/x^2}} = \frac{1}{\sqrt{1+0+0}} = 1$$

and

$$\lim_{x\to -\infty} \frac{x}{\sqrt{x^2+3x+2}} = \lim_{x\to -\infty} \frac{1}{\sqrt{x^2+3x+2}/x}$$
$$= \lim_{x\to -\infty} -\frac{1}{\sqrt{(x^2+3x+2)/x^2}} = \lim_{x\to -\infty} -\frac{1}{\sqrt{1+3/x+2/x^2}} = -1$$

the curve has two horizontal asymptotes $y = 1$ and $y = -1$.

7. (a) Since $(x^2-1)/(x^3-1)$ is continuous at 2 (it is a rational function so it is continuous in its domain)

$$\lim_{x\to 2} \frac{x^2-1}{x^3-1} = \frac{2^2-1}{2^3-1} = \frac{3}{7}$$

(b)

$$\lim_{x\to 1} \frac{x^2-1}{x^3-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)(x^2+x+1)}$$
$$= \lim_{x\to 1} \frac{x+1}{x^2+x+1} = \frac{2}{3}$$

(c)

$$\lim_{x\to\infty} \frac{x^2-1}{x^3-1} = \lim_{x\to\infty} \frac{(x^2-1)/x^3}{(x^3-1)/x^3}$$
$$= \lim_{x\to\infty} \frac{1/x - 1/x^3}{1 - 1/x^3} = \frac{0-0}{1-0} = 0$$