(1) No books, notes or calculators are allowed.
(2) Show your work in details.

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You may need the following formulas: The curvature and torsion of a smooth curve $\gamma : I \rightarrow \mathbb{R}^3$ are given by

$$\kappa = \frac{||\gamma' \wedge \gamma''||}{||\gamma'||^3}$$

$$\tau = -\frac{\det \begin{bmatrix} \gamma' & \gamma'' & \gamma''' \end{bmatrix}}{||\gamma' \wedge \gamma''||^2}$$

(1) (40 points) Let $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$ be the curve $\gamma(t) = (\cos t, \sin t, 2t)$.

(a) (20 points) Find the curvature and torsion of $\gamma$. 

(b) (10 points) Let $T, N$ and $B$ be the unit tangent, normal and binormal vectors of $\gamma$, respectively, and let

$$T' = \frac{dT}{dt}, \quad N' = \frac{dN}{dt} \quad \text{and} \quad B' = \frac{dB}{dt}.$$ 

Write down the Frenet-Serret equations relating \{T, N, B\} and \{T', N', B'\}.
(c) (10 points) Express

\[ T'' = \frac{d^2T}{dt^2}, \quad N'' = \frac{d^2N}{dt^2} \quad \text{and} \quad B'' = \frac{d^2B}{dt^2} \]

as linear combinations of \( T, N \) and \( B \).
(2) (30 points) Do the following:

(a) (15 points) Find the minimum and maximum values of the curvature of the ellipse

\[ C = \left\{ (x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\} \subset \mathbb{R}^2 \]

where \( a \) and \( b \) are constants satisfying \( a \geq b > 0 \).
(b) (15 points) Show that two ellipses

\[ C_1 = \left\{ (x, y) : \frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1 \right\} \]

\[ C_2 = \left\{ (x, y) : \frac{x^2}{a_2^2} + \frac{y^2}{b_2^2} = 1 \right\} \]

in \( \mathbb{R}^2 \) are congruent if and only if either \( a_1 = a_2, b_1 = b_2 \) or \( a_1 = b_2, b_1 = a_2 \), where \( a_i \) and \( b_i \) are positive constants for \( i = 1, 2 \).
(3) (30 points) Let $\gamma: I \to \mathbb{R}^2$ be a regular smooth curve parameterized by arc length with nowhere vanishing curvature $\kappa(s)$ on the open interval $I$. A circular disk of radius 1 rolls without slipping along $\gamma(I)$:

(a) (15 points) Find a curve $\alpha: I \to \mathbb{R}^2$ parameterizing the trajectory of a fixed point on the boundary of the disk. Express your answer in terms of $\gamma, \gamma', \gamma''$ and $\kappa$. 
(b) (15 points) Find the curvature of $\alpha$. Express your answer in terms of $\kappa$, $\kappa'$ and $\int \kappa ds$. For simplicity, you may assume that $\det \left[ \begin{array}{cc} \gamma' & \gamma'' \end{array} \right] > 0$. 