Solutions for Math 348 Assignment #5

(1) Let $f(x)$ be a smooth function on an open interval $I$ satisfying $f(x) > 0$ for all $x \in I$ and let $S$ be the surface in the $xyz$-space obtained by rotating the curve 
\[
\{(x, y, z) : y - f(x) = z = 0\}
\]
around the $x$-axis. Show that $S$ is a regular surface.

**Proof.** The surface $S$ is given by 
\[
S = \{(x, y, z) : f(x) - \sqrt{y^2 + z^2} = 0, \ x \in I\}
\]
\[
= \{(x, y, z) : (f(x))^2 - y^2 - z^2 = 0, \ x \in I\}.
\]
Let $F(x, y, z) = (f(x))^2 - y^2 - z^2$. Since 
\[
\{(x, y, z) : \partial F/\partial x = \partial F/\partial y = \partial F/\partial z = F(x, y, z) = 0, \ x \in I\}
\]
\[
= \{(x, y, z) : f'(x)f(x) = y = z = (f(x))^2 - y^2 - z^2 = 0, \ x \in I\}
\]
\[
= \{(x, y, z) : f(x) = y = z = 0, \ x \in I\} = \emptyset,
\]
$S$ is a regular surface. \(\square\)

(2) Let $F_n = \{x^n + y^n + z^n = 1\} \subset \mathbb{R}^3$ for a positive integer $n$. Show that $F_n$ is a connected regular surface for all $n$.

**Proof.** Let $f(x, y, z) = x^n + y^n + z^n - 1$. Since 
\[
\{(x, y, z) : \partial f/\partial x = \partial f/\partial y = \partial f/\partial z = f(x, y, z) = 0\}
\]
\[
= \{(x, y, z) : nx^{n-1} = ny^{n-1} = nz^{n-1} = f(x, y, z) = 0\}
\]
\[
= \{(x, y, z) : x = y = z = x^n + y^n + z^n - 1 = 0\} = \emptyset,
\]
$F_n = \{f(x, y, z) = 0\}$ is a regular surface. It remains to show that $F_n$ is connected.

For $n$ odd, $F_n$ can be parameterized by $\varphi : \mathbb{R}^2 \to F_n$ where 
\[
\varphi(u, v) = (u, v, \sqrt[1-n]{u^n - v^n}).
\]
Since $\varphi$ is continuous and $\mathbb{R}^2$ is connected, $F_n = \varphi(\mathbb{R}^n)$ is connected.

\(^1\)http://www.math.ualberta.ca/~xichen/math34815f/hw5sol.pdf
For $n$ even, $F_n$ can be locally parameterized by $\varphi_i : U \to F_n$ where $U = \{(u, v) : u^n + v^n < 1\} \subset \mathbb{R}^2$ and
\[
\begin{align*}
\varphi_1(u, v) &= (\sqrt{1 - u^n - v^n}, u, v) \\
\varphi_2(u, v) &= (u, \sqrt{1 - u^n - v^n}, v) \\
\varphi_3(u, v) &= (u, v, \sqrt{1 - u^n - v^n}) \\
\varphi_4(u, v) &= (-\sqrt{1 - u^n - v^n}, u, v) \\
\varphi_5(u, v) &= (u, -\sqrt{1 - u^n - v^n}, v) \\
\varphi_6(u, v) &= (u, v, -\sqrt{1 - u^n - v^n}).
\end{align*}
\]

Clearly,
\[
\bigcup_{i=1}^6 \varphi_i(U) = F_n.
\]

First, we show that $U$ is connected.

Let $p = (0, 0) \in U$. For every $q \in U$, the line segment
\[
\overline{pq} = \{tq : 0 \leq t \leq 1\} \subset U
\]
and hence $U$ is connected. And since $\varphi_i$ is continuous, $\varphi_i(U)$ is connected for all $i$.

Since
\[
(3^{-1/n}, 3^{-1/n}, 3^{-1/n}) \in \varphi_1(U) \cap \varphi_2(U) \cap \varphi_3(U)
\]
$\varphi_1(U) \cap \varphi_2(U) \cap \varphi_3(U) \neq \emptyset$. Therefore, $\varphi_1(U) \cup \varphi_2(U) \cup \varphi_3(U)$ is connected.

Since
\[
(-3^{-1/n}, -3^{-1/n}, -3^{-1/n}) \in \varphi_4(U) \cap \varphi_5(U) \cap \varphi_6(U)
\]
$\varphi_4(U) \cap \varphi_5(U) \cap \varphi_6(U) \neq \emptyset$. Therefore, $\varphi_4(U) \cup \varphi_5(U) \cup \varphi_6(U)$ is connected.

And since
\[
(-3^{-1/n}, 3^{-1/n}, 3^{-1/n}) \in \varphi_3(U) \cap \varphi_4(U) \Rightarrow \varphi_3(U) \cap \varphi_4(U) \neq \emptyset
\]
we conclude that $F_n = \cup \varphi_i(U)$ is connected. \hfill \Box

(3) (MPDC p. 66 Ex. 7) Let $f(x, y, z) = (x + y + z - 1)^2$.

(a) Locate the critical points and critical values of $f$.

(b) For what values $c$ is the set $\{f(x, y, z) = c\}$ a nonempty regular surface?

(c) Do the same for the function $f(x, y, z) = xyz^2$. 

Solution. For \( f(x, y, z) = (x + y + z - 1)^2 \),

\[
J_f = 2(x + y + z - 1) \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}
\]

Therefore, \( \text{rank}(J_f) < 1 \) if and only if \( x + y + z - 1 = 0 \). Hence \( \{(x + y + z - 1)^2 = 0\} \) is the set of critical points of \( f \) and 0 is the critical value of \( f \).

For \( S = \{(x + y + z - 1)^2 = c\} \) to be nonempty, we must have \( c \geq 0 \).

For \( c = 0 \), \( S = \{x + y + z - 1 = 0\} \) is a hyperplane and hence a regular surface.

For \( c > 0 \), since \( c \) is not a critical value of \( f \), \( S \) is a regular surface.

In conclusion, \( S = \{(x + y + z - 1)^2 = c\} \) is a nonempty regular surface if and only if \( c \geq 0 \).

For \( f(x, y, z) = xyz^2 \),

\[
J_f = \begin{bmatrix} yz^2 & xz^2 & 2xyz \\ xz^2 & 2xyz & 2xy \end{bmatrix} = z \begin{bmatrix} yz & xz & 2xy \end{bmatrix}
\]

Therefore, \( \text{rank}(J_f) < 1 \) if and only if \( z = 0 \) or \( x = y = 0 \). The set \( \{z = 0\} \cup \{x = y = 0\} \) is the set of critical points of \( f \) and 0 is the critical value of \( f \).

Note that \( S = \{xyz^2 = c\} \) is nonempty for all \( c \). If \( c \neq 0 \), \( S \) is a regular surface since \( c \) is not a critical value of \( f \). When \( c = 0 \), we claim that \( S = \{xyz^2 = 0\} \) is not a regular surface. If it is, there must exist an open set \( V \subset \mathbb{R}^3 \) containing \( p = (0, 0, 0) \) such that one of the projections

\[
\begin{align*}
\pi_1 : S \cap V &\to \mathbb{R}^2 \text{ by } \pi_1(x, y, z) = (y, z) \\
\pi_2 : S \cap V &\to \mathbb{R}^2 \text{ by } \pi_2(x, y, z) = (x, z) \\
\pi_3 : S \cap V &\to \mathbb{R}^2 \text{ by } \pi_3(x, y, z) = (x, y)
\end{align*}
\]

is one-to-one. But

\[
\begin{align*}
\pi_1(x, 0, 0) = (0, 0) &\quad \text{for all } x \in \mathbb{R} \Rightarrow \pi_1 \text{ is not one-to-one} \\
\pi_2(0, y, 0) = (0, 0) &\quad \text{for all } y \in \mathbb{R} \Rightarrow \pi_2 \text{ is not one-to-one} \\
\pi_3(0, 0, z) = (0, 0) &\quad \text{for all } z \in \mathbb{R} \Rightarrow \pi_3 \text{ is not one-to-one.}
\end{align*}
\]

Contradiction. So \( S = \{xyz^2 = 0\} \) is not a regular surface. In conclusion, \( S = \{xyz^2 = c\} \) is a nonempty regular surface if and only if \( c \neq 0 \). \( \square \)

(4) Let \( f : \mathbb{R}^m \to \mathbb{R}^l \) and \( g : \mathbb{R}^n \to \mathbb{R}^m \) be two smooth maps. Show that if \( p \in \mathbb{R}^n \) is a critical point of \( f \circ g \), then either \( p \) is a critical point of \( g \) or \( g(p) \) is a critical point of \( f \).
Proof. Suppose that \( p \) is not a critical point of \( g \) and \( g(p) \) is not a critical point of \( f \). Then \( \text{rank}(J_f(q)) = l \) and \( \text{rank}(J_g(p)) = m \) for \( q = f(p) \).

We claim that \( \text{rank}(AB) = l \) for \( l \times m \) matrix \( A \) and \( m \times n \) matrix \( B \) if \( \text{rank}(A) = l \) and \( \text{rank}(B) = m \).

Let \( \varphi_A : \mathbb{R}^m \to \mathbb{R}^l \) and \( \varphi_B : \mathbb{R}^n \to \mathbb{R}^m \) be the linear transformations given by \( \varphi_A(u) = Au \) and \( \varphi_B(v) = Bv \), respectively. Since \( \text{rank}(\varphi_A) = \text{rank}(A) = l \) and \( \text{rank}(\varphi_B) = \text{rank}(B) = m \), both \( \varphi_A \) and \( \varphi_B \) are surjective. It follows that \( \varphi_A \circ \varphi_B = \varphi_{AB} \) is surjective, where \( \varphi_{AB} : \mathbb{R}^n \to \mathbb{R}^l \) is the linear transformation given by \( \varphi_{AB}(x) = ABx \). Therefore,

\[
\text{rank}(\varphi_{AB}) = \text{rank}(AB) = l.
\]

This proves that \( \text{rank}(A) = l \) and \( \text{rank}(B) = m \) \( \Rightarrow \) \( \text{rank}(AB) = l \)

for \( l \times m \) matrix \( A \) and \( m \times n \) matrix \( B \). Therefore,

\[
\text{rank}(J_f(q)) = l \quad \text{and} \quad \text{rank}(J_g(p)) = m
\]

\( \Rightarrow \) \( \text{rank}(J_{f \circ g}(p)) = \text{rank}(J_f(q)J_g(p)) = l \)

and \( p \) is not a critical point of \( f \circ g \). Contradiction. \( \square \)

(5) (MPDC p. 66 Ex. 9) Let \( V \) be an open set in \( \mathbb{R}^2 \). Show that

\[
\{(x, y, z) \in \mathbb{R}^3 : z = 0 \text{ and } (x, y) \in V\}
\]

is a regular surface in \( \mathbb{R}^3 \).

Proof. Let \( f : V \to \mathbb{R}^3 \) be the map \( f(u, v) = (u, v, 0) \). Then \( f \) maps \( V \) bijectively onto

\[
S = \{(x, y, z) \in \mathbb{R}^3 : z = 0 \text{ and } (x, y) \in V\},
\]

\( f \) is \( C^\infty \) and

\[
\text{rank}(J_f) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = 2.
\]

So \( S \) is a regular surface. \( \square \)