Solutions for Math 348 Assignment #3

(1) (MPDC p. 15 Ex. 6) Given two nonparallel planes

\[ a_i x + b_i y + c_i z + d_i = 0 \]

in \( \mathbb{R}^3 \) for \( i = 1, 2 \), show that their line \( L \) of intersection may be parameterized by

\[ x = x_0 + u_1 t, \quad y = y_0 + u_2 t, \quad z = z_0 + u_3 t \]

where \( (x_0, y_0, z_0) \in L \) and \( \mathbf{u} = (u_1, u_2, u_3) \) is the cross product \( \mathbf{u} = \mathbf{v}_1 \times \mathbf{v}_2 \) for \( \mathbf{v}_i = (a_i, b_i, c_i) \).

Proof. Let \( \gamma(t) = (x_0 + u_1 t, y_0 + u_2 t, z_0 + u_3 t) = \mathbf{u}_0 + t \mathbf{u} \), where \( \mathbf{u}_0 = (x_0, y_0, z_0) \). It suffices to show that

\[ \mathbf{v}_i \cdot \gamma(t) + d_i = 0 \]

for \( i = 1, 2 \).

Since \( \mathbf{v}_i \cdot \mathbf{u} = \mathbf{v}_i \cdot (\mathbf{v}_1 \times \mathbf{v}_2) = 0 \),

\[ \mathbf{v}_i \cdot \gamma(t) = \mathbf{v}_i \cdot \mathbf{u}_0 + t \mathbf{v}_i \cdot \mathbf{u} = \mathbf{v}_i \cdot \mathbf{u}_0. \]

And since \( \mathbf{u}_0 \in L \), \( \mathbf{v}_i \cdot \mathbf{u}_0 + d_i = 0 \) and hence

\[ \mathbf{v}_i \cdot \gamma(t) + d_i = \mathbf{v}_i \cdot \mathbf{u}_0 + d_i = 0 \]

for \( i = 1, 2 \). \( \square \)

(2) Let \( V \) be a finite-dimensional vector space over \( \mathbb{R} \).

(a) Let \( \dim V = 3 \). Show that

- For \( v \in V \), \( v = 0 \) if and only if \( v \wedge w = 0 \) for all \( w \in \wedge^2 V \).
- For \( w \in \wedge^2 V \), \( w = 0 \) if and only if \( w \wedge v = 0 \) for all \( v \in V \).

(b) Are the above statements true for \( \dim V \geq 3 \)? Justify your answer.

Proof. This holds for all \( V \) with \( \dim V \geq 3 \). Let \( \{e_1, e_2, ..., e_n\} \) be a basis of \( V \).

If \( v = 0 \), then \( v \wedge w = 0 \) for all \( w \in \wedge^2 V \).

Suppose that \( v \wedge w = 0 \) for all \( w \in \wedge^2 V \). Let

\[ v = \sum_{i=1}^{n} a_i e_i. \]

\[ \text{http://www.math.ualberta.ca/~xichen/math34815f/hw3sol.pdf} \]
Then
\[ v \wedge (e_1 \wedge e_2) = 0 \Rightarrow \sum_{i=1}^{n} a_i e_i \wedge e_1 \wedge e_2 = 0 \]
\[ \Rightarrow \sum_{i=3}^{n} a_i e_i \wedge e_1 \wedge e_2 = 0 \]
\[ \Rightarrow a_3 = a_4 = ... = a_n = 0 \]
\[ v \wedge (e_1 \wedge e_3) = 0 \Rightarrow \sum_{i=1}^{n} a_i e_i \wedge e_1 \wedge e_3 = 0 \]
\[ \Rightarrow a_2 e_2 \wedge e_1 \wedge e_3 + \sum_{i=4}^{n} a_i e_i \wedge e_1 \wedge e_3 = 0 \]
\[ \Rightarrow a_2 = a_4 = ... = a_n = 0 \]
\[ v \wedge (e_2 \wedge e_3) = 0 \Rightarrow \sum_{i=1}^{n} a_i e_i \wedge e_2 \wedge e_3 = 0 \]
\[ \Rightarrow a_1 e_1 \wedge e_2 \wedge e_3 + \sum_{i=4}^{n} a_i e_i \wedge e_2 \wedge e_3 = 0 \]
\[ \Rightarrow a_1 = a_4 = ... = a_n = 0. \]
Therefore, \( a_1 = a_2 = ... = a_n = 0 \) and \( v = 0 \).

If \( w = 0 \in \wedge^2 V \), then \( v \wedge w = 0 \) for all \( v \in V \).

Suppose that \( v \wedge w = 0 \) for all \( v \in V \). Let
\[ w = \sum_{1 \leq i < j \leq n} a_{ij} e_i \wedge e_j. \]
Then
\[ e_1 \wedge w = 0 \Rightarrow \sum_{1 \leq i < j \leq n} a_{ij} e_1 e_i \wedge e_j = 0 \]
\[ \Rightarrow a_{ij} = 0 \text{ for all } i, j \neq 1 \]
\[ e_2 \wedge w = 0 \Rightarrow \sum_{1 \leq i < j \leq n} a_{ij} e_2 e_i \wedge e_j = 0 \]
\[ \Rightarrow a_{ij} = 0 \text{ for all } i, j \neq 2 \]
\[ e_3 \wedge w = 0 \Rightarrow \sum_{1 \leq i < j \leq n} a_{ij} e_3 e_i \wedge e_j = 0 \]
\[ \Rightarrow a_{ij} = 0 \text{ for all } i, j \neq 3 \]
Therefore,
\[ a_{ij} = 0 \text{ if } 1 \not\in \{i, j\}, 2 \not\in \{i, j\} \text{ or } 3 \not\in \{i, j\}. \]
And for all pairs \((i, j)\), we cannot have \(\{1, 2, 3\} \subset \{i, j\}\). Therefore, \(a_{ij} = 0\) for all \(i, j\) and \(w = 0\).  

\(\square\)

(3) (MPDC p. 16 Ex. 11) Let 
\[ P = \{t_1 \mathbf{u} + t_2 \mathbf{v} + t_3 \mathbf{w} : 0 \leq t_1, t_2, t_3 \leq 1\} \]
be the parallelepiped generated by the three linearly independent vectors \(\mathbf{u}, \mathbf{v}, \mathbf{w}\) in \(\mathbb{R}^3\) and let \(V\) be the volume of \(P\). Show that

\[ V^2 = ||\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}|| \]

**Proof.** The volume \(V\) of \(P\) is given by the integral

\[ V = \int_P dxdydz. \]
Consider the map \(f : \mathbb{R}^3 \to \mathbb{R}^3\) given by

\[ f(t_1, t_2, t_3) = t_1 \mathbf{u} + t_2 \mathbf{v} + t_3 \mathbf{w}. \]
Then \(f\) is one-to-one and onto and \(f(D) = P\), where \(D\) is the unit cube 
\[ D = \{(t_1, t_2, t_3) : 0 \leq t_1, t_2, t_3 \leq 1\}. \]
The Jacobian of \(f\) is exactly

\[ J_f = \begin{bmatrix} \frac{\partial f}{\partial t_1} & \frac{\partial f}{\partial t_2} & \frac{\partial f}{\partial t_3} \end{bmatrix} = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}. \]
Therefore, by substitution rule for integration, we have

\[ V = \int_P dxdydz = \int_D |\det(J_f)|dxdydz \]
\[ = |\det(J_f)| \int_0^1 \int_0^1 \int_0^1 dxdydz = |\det(J_f)| \]
\[ = |\det \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}| = ||\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}|| \]
and

\[ V^2 = ||\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}||^2 \]
\[ = (\det \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix})^2 \]
\[ = \det \left( \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}^T \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix} \right) \]
\[ = \det \left( \begin{bmatrix} \mathbf{u}^T & \mathbf{v}^T & \mathbf{w}^T \end{bmatrix} \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix} \right) \]
(4) Let $V$ be a finite-dimensional vector space over $\mathbb{R}$ and $v \in V$ be a nonzero vector in $V$.

(a) Let $\dim V = 3$. For $w \in \bigwedge^2 V$, show that $v \wedge w = 0$ if and only if there exists $u \in V$ such that $w = v \wedge u$.

(b) Is the above statement true for $\dim V \geq 3$? Justify your answer.

Proof. This holds for all $V$ with $\dim V \geq 3$.

Let $v_1 = v$. We expand $\{v_1\}$ to a basis $\{v_1, v_2, \ldots, v_n\}$ of $V$.

If $w = v \wedge u$, then $v \wedge w = (v \wedge v) \wedge u = 0$.

Suppose that $v \wedge w = 0$. Let

$$w = \sum_{1 \leq i < j \leq n} a_{ij} v_i \wedge v_j.$$ 

Then

$$v \wedge w = 0 \Rightarrow \sum_{1 \leq i < j \leq n} a_{ij} v_1 \wedge v_i \wedge v_j = 0 \Rightarrow a_{ij} = 0 \text{ for all } 2 \leq i < j \leq n.$$ 

So

$$w = \sum_{1 \leq i < j \leq n} a_{ij} v_i \wedge v_j = \sum_{j=2}^n a_{1j} v_1 \wedge v_j$$ 

$$= v_1 \wedge \sum_{j=2}^n a_{1j} v_j = v \wedge u.$$ 

(5) Find the binormal vectors of the curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ given by $\gamma(t) = (t, t^2, t^3)$ for all $t$.

Solution. A binormal vector of $\gamma$ is given by

$$\gamma'(t) \wedge \gamma''(t) = (1, 2t, 3t^2) \wedge (0, 2, 6t)$$ 

$$= 2e_1 \wedge e_2 + 6t^2 e_2 \wedge e_3 + 6te_1 \wedge e_3$$ 

$$= 6t^2 e_1 - 6te_2 + 2e_3.$$
and the unit binormal vector is

\[
\frac{1}{\sqrt{9t^4 + 9t^2 + 1}} (3t^2 e_1 - 3t e_2 + e_3).
\]

□