Solutions for Math 348 Assignment #2

(1) Let $\alpha : I \to \mathbb{R}^n$ and $\beta : I \to \mathbb{R}^n$ be two $C^m$-curves. Show that

$$\frac{d^m}{dt^m} (\alpha(t), \beta(t)) = \sum_{j=0}^{m} \binom{m}{j} \alpha^{(j)}(t), \beta^{(m-j)}(t).$$

Proof. We prove it by induction on $m$.

When $m = 1$,

$$\frac{d}{dt} (\alpha(t), \beta(t)) = \alpha(t), \beta'(t) + \alpha'(t), \beta(t)$$

$$= \sum_{j=0}^{1} \binom{1}{j} \alpha^{(j)}(t), \beta^{(1-j)}(t).$$

Suppose that the equality holds for $m = k$. That is,

$$\frac{d^k}{dt^k} (\alpha(t), \beta(t)) = \sum_{j=0}^{k} \binom{k}{j} \alpha^{(j)}(t), \beta^{(k-j)}(t).$$

Then

$$\frac{d^{k+1}}{dt^{k+1}} (\alpha(t), \beta(t)) = \frac{d}{dt} \left( \frac{d^k}{dt^k} (\alpha(t), \beta(t)) \right)$$

$$= \frac{d}{dt} \left( \sum_{j=0}^{k} \binom{k}{j} \alpha^{(j)}(t), \beta^{(k-j)}(t) \right)$$

$$= \sum_{j=0}^{k} \binom{k}{j} \left( \alpha^{(j)}(t), \beta^{(k-j+1)}(t) + \alpha^{(j+1)}(t), \beta^{(k-j)}(t) \right)$$

$$= \sum_{j=0}^{k} \binom{k}{j} \alpha^{(j)}(t), \beta^{(k-j+1)}(t) + \sum_{j=0}^{k} \binom{k}{j} \alpha^{(j+1)}(t), \beta^{(k-j)}(t)$$

$$= \sum_{j=0}^{k} \binom{k}{j} \alpha^{(j)}(t), \beta^{(k-j+1)}(t) + \sum_{j=1}^{k+1} \binom{k}{j-1} \alpha^{(j)}(t), \beta^{(k-j+1)}(t)$$

$$= \sum_{j=0}^{k+1} \binom{k}{j} \alpha^{(j)}(t), \beta^{(k-j+1)}(t) + \sum_{j=0}^{k+1} \binom{k}{j-1} \alpha^{(j)}(t), \beta^{(k-j+1)}(t)$$

$$= \sum_{j=0}^{k+1} \left( \binom{k}{j} + \binom{k}{j-1} \right) \alpha^{(j)}(t), \beta^{(k-j+1)}(t)$$

1http://www.math.ualberta.ca/~xichen/math34815f/hw2sol.pdf
\[
= \sum_{j=0}^{k+1} \binom{k+1}{j} \alpha_{(j)} \beta_{(k+1-j)}(t)
\]

since

\[
\binom{k}{j} + \binom{k}{j-1} = \binom{k+1}{j}.
\]

So the equality also holds for \( m = k + 1 \). \( \square \)

(2) A regular smooth curve \( \gamma : I \to \mathbb{R}^n \) has zero curvature on \( I \) if and only if it is a line.

**Proof.** Suppose that \( \gamma \) is parameterize by its arc length. Then \( \kappa(s) \equiv 0 \) if and only if \( \gamma''(s) \equiv 0 \). That is,

\[
\kappa(s) \equiv 0 \iff \gamma''(s) \equiv 0
\]

\[
\iff \gamma'(s) \equiv b \text{ for some } b \in \mathbb{R}^n
\]

\[
\iff \gamma'(s) \equiv a + s b \text{ for some } a, b \in \mathbb{R}^n
\]

\[
\iff \gamma(I) \text{ is a line in } \mathbb{R}^n.
\]

\( \square \)

(3) A regular smooth curve \( \gamma : I \to \mathbb{R}^2 \) has nonzero constant curvature on \( I \) if and only if it is an arc of a circle.

**Proof.** Suppose that \( \gamma(s) = (x(s), y(s)) \) is parameterized by its arc length. Then \( \kappa(s) \equiv k \) is an constant if and only if \( ||\gamma''(s)|| \equiv k \). And since \( \gamma'(s) \) and \( \gamma''(s) \) are orthogonal and \( ||\gamma'(s)|| \equiv 1 \), \( \kappa(s) \equiv k \) if and only if

\[
\begin{align*}
\begin{cases}
  kx'(s) = y''(s) \\
k y'(s) = -x''(s)
\end{cases}
\text{ or } \begin{cases}
  kx'(s) = -y''(s) \\
k y'(s) = x''(s)
\end{cases}
\end{align*}
\]
Therefore,
\[ \kappa(s) \equiv k \neq 0 \iff \begin{cases} x'' + k^2 x' = 0 \\ y' = \pm \frac{1}{k} x'' \end{cases} \]
\[ \iff \begin{cases} x' = c_1 \cos(ks) + c_2 \sin(ks) \\ y' = \pm(-c_1 \sin(ks) + c_2 \cos(ks)) \end{cases} \]
\[ \iff \begin{cases} x = \frac{c_1}{k} \sin(ks) - \frac{c_2}{k} \cos(ks) + x_0 \\ y = \pm \left( \frac{c_1}{k} \cos(ks) + \frac{c_2}{k} \sin(ks) \right) + y_0 \end{cases} \]
\[ \iff (x - x_0)^2 + (y - y_0)^2 = \frac{c_1^2 + c_2^2}{k^2} \]
for some constants \( c_1, c_2, x_0, y_0 \in \mathbb{R} \). \( \square \)

(4) (MPDC p. 25 Ex. 11) For a smooth regular plane curve given by \( \rho = \rho(\theta) \) under the polar coordinates \( (\rho, \theta) \), do the following:
(a) Show that its arc length is
\[ \int_a^b \sqrt{(\rho(\theta))^2 + (\rho'(\theta))^2} d\theta \]
from \( \theta = a \) to \( \theta = b \).
(b) Show that its curvature is
\[ \kappa(\theta) = \frac{|2(\rho')^2 - \rho \rho'' + \rho^2|}{((\rho')^2 + \rho^2)^{3/2}}. \]

**Proof.** The curve is parameterized by
\[ \gamma(\theta) = (\rho(\theta) \cos \theta, \rho(\theta) \sin \theta). \]
So the arc length of \( \gamma \) on \([a, b]\) is
\[ \int_a^b ||\gamma'(\theta)|| d\theta = \int_a^b \sqrt{((\rho(\theta) \cos \theta)')^2 + ((\rho(\theta) \sin \theta)')^2} d\theta \]
where \( \gamma' = ((\rho \cos \theta)', (\rho \sin \theta)') = (\rho' \cos \theta - \rho \sin \theta, \rho' \sin \theta + \rho \cos \theta) \)
\[ ||\gamma'|| = (\rho' \cos \theta - \rho \sin \theta)^2 + (\rho' \sin \theta + \rho \cos \theta)^2 \]
\[ = (\rho')^2(\cos \theta)^2 - 2\rho' \rho \cos \theta \sin \theta + \rho^2(\sin \theta)^2 \]
\[ + (\rho')^2(\sin \theta)^2 + 2\rho' \rho \cos \theta \sin \theta + \rho^2(\cos \theta)^2 \]
\[ = \rho^2 + (\rho')^2. \]
Hence
\[
\gamma'' = ((\rho' \cos \theta - \rho \sin \theta)', (\rho' \sin \theta + \rho \cos \theta)')
\]
\[
= (\rho'' \cos \theta - 2\rho' \sin \theta - \rho \cos \theta, \rho'' \sin \theta + 2\rho' \cos \theta - \rho \sin \theta)
\]
\[
\gamma' \wedge \gamma'' = ((\rho' \cos \theta - \rho \sin \theta)(\rho'' \sin \theta + 2\rho' \cos \theta - \rho \sin \theta)
\]
\[
- (\rho' \sin \theta + \rho \cos \theta)(\rho'' \cos \theta - 2\rho' \sin \theta - \rho \cos \theta))e_1 \wedge e_2
\]
\[
= (2(\rho')^2 - \rho \rho'' + \rho^2)e_1 \wedge e_2
\]

The curvature of $\gamma$ is then given by
\[
\kappa = \frac{||\gamma' \wedge \gamma''||}{||\gamma'||^3} = \frac{|2(\rho')^2 - \rho \rho'' + \rho^2|}{(\rho^2 + (\rho')^2)^{3/2}}.
\]

(5) Find the curvatures of the curve
\[
\{x^2 + y^2 - z^2 = x + y + z + 2 = 0\} \subset \mathbb{R}^3
\]

at the points $(0, -1, -1)$ and $(-3, -4, 5)$, respectively.

**Solution.** The curve can be parameterized by
\[
\begin{align*}
\{ & x^2 + y^2 - z^2 = 0 \implies \begin{cases} x^2 + y^2 = z^2 \\ -(x + y + 2) = z \end{cases} \\
& x + y + z + 2 = 0 \implies \begin{cases} xy + 2x + 2y + 2 = 0 \\ z = -(x + y + 2) \end{cases} \\
& \implies \begin{cases} x = t \\ y = -2 + \frac{2}{t + 2} \\ z = -t - \frac{2}{t + 2} \end{cases} \\
& \implies \gamma(t) = \left(t, -2 + \frac{2}{t + 2}, -t - \frac{2}{t + 2}\right).
\end{align*}
\]

So
\[
\gamma' = \left(1, -\frac{2}{(t + 2)^2}, -1 + \frac{2}{(t + 2)^2}\right)
\]
\[
\gamma'' = \left(0, \frac{4}{(t + 2)^3}, -\frac{4}{(t + 2)^3}\right)
\]
and the curvature of $\gamma$ is
$$\kappa(t) = \frac{||\gamma' \wedge \gamma''||}{||\gamma'||^3} = \frac{\sqrt{6}|t + 2|^3}{((t + 2)^4 - 2(t + 2)^2 + 4)^{3/2}}.$$ 
Hence the curvatures of $\gamma$ at $(0, -1, -1)$ and $(-3, -4, 5)$ are
$$\gamma'(0) = \frac{\sqrt{2}}{3} \text{ and } \gamma'(-3) = \frac{\sqrt{2}}{3},$$
respectively. $\square$