1 (20) | 2 (20) | 3 (20) | 4 (20) | 5 (20) | Total (100)
You may need the following formulas: The curvature and torsion of a smooth curve \( \gamma : I \to \mathbb{R}^3 \) are given by

\[
\kappa = \frac{||\gamma' \wedge \gamma''||}{||\gamma'||^3}
\]

\[
\tau = -\frac{\det [\gamma' \ \gamma'' \ \gamma''']}{||\gamma' \wedge \gamma''||^2}
\]
(1) (20 pts) Find the curvature and torsion of the curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ given by $\gamma(t) = (t, t^2, t^3)$ and write down the Frenet equations associated to $\gamma$. 
(2) (20 pts) Do the following:

(a) Show that the intersection of two spheres

\[ S_1 = \{(x, y, z) : (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r_1^2\} \]
\[ S_2 = \{(x, y, z) : (x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2 = r_2^2\} \]

in \( \mathbb{R}^3 \) is a curve with constant positive curvature and zero torsion, assuming that \( S_1 \) and \( S_2 \) meet at more than one point and \( S_1 \neq S_2 \).
(b) Show that a smooth curve $\gamma : I \to \mathbb{R}^3$ has constant positive curvature and zero torsion if and only if $\gamma(I)$ is contained in a circle.
(3) (20 pts) Let \( f(x, y, z) = (x + y + z - 1)^2 \).
   
   (a) Locate the critical points and critical values of \( f \).
   
   (b) For what values \( c \) is the set \( \{ f(x, y, z) = c \} \) a nonempty regular surface?
   
   (c) Do the same for the function \( f(x, y, z) = xyz^2 \).
(4) (20 pts) Let $S$ be a regular surface in $\mathbb{R}^3$ and let $f : S \to \mathbb{R}$ be the map given by $f(p) = ||p - p_0||^2$ for a fixed point $p_0 \in \mathbb{R}^3$. Show that $df_p(w) = 2w.(p - p_0)$ for all $p \in S$ and $w \in T_{S,p}$. 
(5) (20 pts) Let \( \alpha : I \rightarrow \mathbb{R}^2 \) be a regular smooth curve and \( p \) be a fixed point in \( \mathbb{R}^2 \). Suppose that \( ||\alpha(t) - p|| \) achieves a local maximum at a point \( t_0 \in I \). Show that

\[
\kappa(t_0) \geq \frac{1}{||\alpha(t_0) - p||}.
\]
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