Since $S$ is convex, $\text{Int}(S)$ is convex. And since $\{x_1, x_2, \ldots, x_n\} \subset \text{Int}(S)$, $\text{conv}\{x_1, x_2, \ldots, x_n\} \subset \text{Int}(S)$.

(2) Since every convex set in $\mathbb{R}^1$ is an interval, $\text{conv}(S)$ could be $[a, b]$, $[a, b)$, $(a, b)$ or $(a, b]$, where $a$ and $b$ could be $\pm \infty$. Suppose that $\text{conv}(S)$ is not closed. Then it could be one of the following:

(a) $[a, b)$ with $-\infty < a < b < \infty$;
(b) $(a, b]$ with $-\infty < a < b < \infty$;
(c) $(a, b)$ with $a < b$ and either $a \neq -\infty$ or $b \neq \infty$.

Suppose that we are in the first case that $\text{conv}(S) = [a, b)$. Since $b \notin S$, $b \in S^c$, where $S^c$ is open for $S$ is closed. Therefore, there exists $r > 0$ such that $B(b, r) = (b - r, b + r) \subset S^c$, i.e., $(b - r, b + r) \cap S = \emptyset$. And since $S \subset \text{conv}(S) = [a, b)$, $S \subset [a, b) \setminus (b - r, b + r) = [a, b - r]$. Since $[a, b - r]$ is convex, $\text{conv}(S) \subset [a, b - r]$. Contradiction.

The other two cases can be dealt with similarly.

(3) We first prove the theorem for $S_i$ closed. Let $S_i = [a_i, b_i]$ where $a_i$ or $b_i$ could be $\pm \infty$. Then $S_i \cap S_j \neq \emptyset$ if and only if $\max\{a_i, a_j\} \leq \min\{b_i, b_j\}$. Therefore, $\max\{a_i, a_j\} \leq \min\{b_i, b_j\}$ for any $i \neq j$. Consequently,

$$\max\{a_1, a_2, \ldots, a_m\} \leq \min\{b_1, b_2, \ldots, b_m\}$$

Let $a = \max\{a_1, a_2, \ldots, a_m\}$ and $b = \min\{b_1, b_2, \ldots, b_m\}$. It is not hard to see that $S_1 \cap S_2 \cap \ldots \cap S_m = [a, b]$. Of course, $a \leq b$ implies that $S_1 \cap S_2 \cap \ldots \cap S_m \neq \emptyset$.

Now let us treat arbitrary $S_i$. Let $T_i = \text{cl}(S_i) = [a_i, b_i]$. Since $S_i \cap S_j \neq \emptyset$, $T_i \cap T_j \neq \emptyset$. Therefore, $\cap T_i = [a, b] \neq \emptyset$. Let $c = (a + b)/2$. If $c \in S_i$ for all $i$, we are done. If not, suppose that $c \notin S_i$ for some $i$. Note that $[a, b] \subset [a_i, b_i]$ and $S_i$ could be one of $(a_i, b_i)$, $(a_i, b_i)$, $[a_i, b_i)$ and $[a_i, b_i]$. Therefore, $c \notin S_i$ only if $a = b = c = a_i$ or $b_i$. Suppose that $a = b = c = a_i$. Since $a_i \notin S_i$, we necessarily have $b_i > a_i$; otherwise, $S_i = \emptyset$. Since $b = \min\{b_1, b_2, \ldots, b_m\}$, there exists $j$ such that $b_j = b$. Since $b_i > a_i = b = b_j$, $i \neq j$. On the other hand, $S_i \cap S_j \subset T_i \cap T_j = \{a_i\}$. But $a_i \notin S_i$ and hence $S_i \cap S_j = \emptyset$. Contradiction. The case $a = b = c = b_i$ follows from a similar argument.

(4) Any $n + 2$ vectors in $\mathbb{R}^n$ are affinely dependent. Therefore, $x, x_1, x_2, \ldots, x_{n+1}$ are affinely dependent, i.e., there exist $\lambda, \lambda_1, \lambda_2, \ldots, \lambda_{n+1}$, not all zero, such that

$$\lambda x + \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_{n+1} x_{n+1} = 0$$

and

$$\lambda + \lambda_1 + \lambda_2 + \ldots + \lambda_{n+1} = 0$$

Note that $\lambda \neq 0$; otherwise, $x_1, x_2, \ldots, x_{n+1}$ will be affinely dependent. Therefore,

$$x = -\sum_{i=1}^{n+1} \left(\frac{\lambda_i}{\lambda}\right) x_i$$

Since $\lambda = -(\lambda_1 + \lambda_2 + \ldots + \lambda_{n+1})$,

$$-\sum_{i=1}^{n+1} \left(\frac{\lambda_i}{\lambda}\right) = 1$$
Therefore, $x$ is an affine combination of $x_1, x_2, \ldots, x_{n+1}$. It remains to show the uniqueness.

Suppose that $x$ can be written as affine combinations of $x_1, x_2, \ldots, x_{n+1}$ in more than one way, i.e.,

$$x = \sum_{i=1}^{n+1} \lambda_i x_i = \sum_{i=1}^{n+1} \lambda'_i x_i$$

with

$$\sum_{i=1}^{n+1} \lambda_i = \sum_{i=1}^{n+1} \lambda'_i = 1$$

and $(\lambda_1, \lambda_2, \ldots, \lambda_{n+1}) \neq (\lambda'_1, \lambda'_2, \ldots, \lambda'_{n+1})$. Then

$$\sum_{i=1}^{n+1} (\lambda_i - \lambda'_i) x_i = 0$$

and

$$\sum_{i=1}^{n+1} (\lambda_i - \lambda'_i) = 0$$

Since $(\lambda_1, \lambda_2, \ldots, \lambda_{n+1}) \neq (\lambda'_1, \lambda'_2, \ldots, \lambda'_{n+1})$, $\lambda_i - \lambda'_i$ are not all zero. Therefore, $x_1, x_2, \ldots, x_{n+1}$ are affinely dependent. Contradiction.

(5) (a) $x = (12/13)x_1 + (3/13)x_2 - (2/13)x_3$ is outside of conv$\{x_1, x_2, x_3\}$.
(b) $x = (8/13)x_1 + (2/13)x_2 + (3/13)x_3$ is inside conv$\{x_1, x_2, x_3\}$.
(c) $x = (2/3)x_1 + (1/3)x_3$ is on the boundary of conv$\{x_1, x_2, x_3\}$.
(d) $x = (9/13)x_1 - (1/13)x_2 + (5/13)x_3$ is outside of conv$\{x_1, x_2, x_3\}$.

(6) (a) Since aff$(S)$ is an affine set, aff$(S)$ is closed. And since $S \subset$ aff$(S)$, cl$(S) \subset$ aff$(S)$.
(b) Since $S \subset$ cl$(S)$, aff$(S) \subset$ aff(cl$(S)$). On the other hand, cl$(S) \subset$ aff$(S)$ by (a) and aff$(S)$ is affine. Consequently, aff(cl$(S)$) $\subset$ aff$(S)$. Therefore, aff(cl$(S)$) = aff$(S)$.
(c) In the last assignment, we have proved that cl(Int$(S)$) = cl$(S)$ if Int$(S) \neq \emptyset$ and $S$ is convex. In case that we only have $S$ is convex, we may restrict $S$ to its affine hull aff$(S)$ $\cong \mathbb{R}^k$. Then the interior of $S$ in aff$(S)$, i.e., relint$(S) \neq \emptyset$. Therefore, cl(relint$(S)$) = cl$(S)$.

Since relint$(S) \subset S$, aff(relint$(S)$) $\subset$ aff$(S)$. On the other hand, we see that cl(relint$(S)$) = cl$(S)$. And by (a), aff(relint$(S)$) $\supset$ cl(relint$(S)$). Therefore, aff(relint$(S)$) $\supset$ cl$(S)$). Since aff(relint$(S)$) is affine, aff(relint$(S)$) $\supset$ aff(cl$(S)$). From (b), aff(cl$(S)$) = aff$(S)$. Therefore, aff(relint$(S)$) $\supset$ aff$(S)$. We are done.