(1) Sketch the following regions in $\mathbb{R}^2$ and determine their convexity (which of them are convex). You must justify your answer.

(a) $S = \{(x, y) : 9x^2 + 9y^2 + 4x - 4y \leq 0\} \subset \mathbb{R}^2$

(b) $S = \{(x, y) : 0 \leq x - y \leq 1, 0 \leq x + y \leq 1\} \subset \mathbb{R}^2$

(c) $S = \{(x, y) : x^2 \leq y \leq x, 0 \leq x \leq 1\} \subset \mathbb{R}^2$

(d) $S = \{(x, y) : x^3 \leq y \leq x^2, 0 \leq x \leq 1\} \subset \mathbb{R}^2$

(e) $S = \{(x, y) : 0 \leq y \leq \sin(x), 0 \leq x \leq \pi\}$

(2) If $A$ and $B$ are convex sets, prove that $A \cap B$ is convex. Give an example that $A$ and $B$ are convex sets but $A \cup B$ is not.

(3) If $A$ and $B$ are convex sets, prove that $A + B$ is convex.

(4) Find the convex hull of the following sets in $\mathbb{R}^2$. You must justify your answer.

(a) $S = \{(x, y) : y = x^2\}$

(b) $S = \{(x, y) : y = x^2, x \geq 0\}$

(c) $S = \{(x, y) : y = x^3\}$

(d) $S = \{(x, y) : xy = 1, x \geq 1/2\}$

(e) $S = \{(x, y) : y = \cos(x)\}$

(5) Prove that the convex hull of a bounded set is bounded.

(6) Let $S$ be a convex set and let $\alpha > 0, \beta > 0$. Prove that $\alpha S + \beta S = (\alpha + \beta)S$ and find an example to show that the convexity of $S$ is necessary.

(7) Let $S$ be a convex set with $\text{Int}(S) \neq \emptyset$.

(a) Prove or give a counterexample: The boundary of $S$ is convex.

(b) Prove or give a counterexample: The boundary of $S$ is not convex.

(c) Suppose that $S$ is compact. Prove that the boundary of $S$ is not convex.

(8) (a) Let $F$ and $G$ be two affine subspaces of $\mathbb{R}^n$. Prove that $F \cup G$ is convex if and only if $F \subset G$ or $G \subset F$.

(b) Show by an example that the union of two arbitrary convex sets may be convex without either of the sets being a subset of the other.

(9) Let $f$ be a linear map from $\mathbb{R}^m \rightarrow \mathbb{R}^n$. Show that $f(S)$ is convex for any convex set $S \subset \mathbb{R}^m$ and $f^{-1}(T)$ is convex for any convex set $T \subset \mathbb{R}^n$.

(10) Let $A$ and $B$ be nonempty sets. Prove the following:

(a) If $A \subset B$, then $A \subset \text{conv}(B)$.

(b) If $A \subset B$ and $B$ is convex, then $\text{conv}(A) \subset B$.

(c) If $A \subset B$, then $\text{conv}(A) \subset \text{conv}(B)$.

(d) $\text{conv}(A) \cup \text{conv}(B) \subset \text{conv}(A \cup B)$.

(e) $\text{conv}(A \cap B) \subset \text{conv}(A) \cap \text{conv}(B)$.

(f) $\text{conv}(\text{conv}(A)) = \text{conv}(A)$.

(g) Find examples to show that equality need not hold in parts (d) and (e).