Math 341 Homework 7 (due Mar. 20)

Problems from the book: (pp. 23-24) 4.4, 4.8, 4.9, 4.10, 4.13

Hints

4.4 Use Caratheodory theorem and Theorem 4.13. We want to show that \( \text{conv}(A) \cap \text{conv}(B) = \emptyset \) if any \( n + 1 \) points of \( B \) can be strictly separated from \( A \) by a hyperplane. If \( \text{conv}(A) \cap \text{conv}(B) \neq \emptyset \), then there is a point \( x \in \text{conv}(A) \cap \text{conv}(B) \). By Caratheodory theorem, \( x \in \text{conv}\{x_1, x_2, \ldots, x_{n+1}\} \) with \( T = \{x_1, x_2, \ldots, x_{n+1}\} \subset B \). Since there is a hyperplane strictly separating \( T \) and \( A \), \( \text{conv}(T) \cap \text{conv}(A) = \emptyset \). Contradiction.

4.8 Use Theorem 4.7. Since \( A \cap \text{int}(B) = \emptyset \) and \( \text{int}(B) = \emptyset \), \( A \) and \( B \) can be separated by a hyperplane \( H = \{ f(x) = \alpha \} \). Suppose that \( A \subset M = \{ f(x) \geq \alpha \} \) and \( B \subset N = \{ f(x) \leq \alpha \} \). So \( \text{bd}(B) \subset N \). Show that \( A \subset H \).

4.9 You need to show that \( \text{cl}(S) \neq \mathbb{R}^n \). If \( \dim(S) < n \), then \( \text{cl}(S) \subset \text{aff}(S) \neq \mathbb{R}^n \). If \( \dim(S) = n \), then \( \text{int}(S) \neq \emptyset \); pick a point \( p \not\in S \) and apply Theorem 4.7 to \( \{p\} \) and \( S \). Show that \( \text{cl}(S) \) is contained in a closed half-space.

4.10 (a) Without the loss of generality, you may assume that \( A \cup B = \mathbb{R}^n \). Show that at least one of \( A \) and \( B \) has nonempty interior. By Theorem 4.7, \( A \) and \( B \) are separated by a hyperplane \( H = \{ f(x) = \alpha \} \). Suppose that \( A \subset M = \{ f(x) \geq \alpha \} \) and \( B \subset N = \{ f(x) \leq \alpha \} \). Use the fact that \( A \cup B = \mathbb{R}^n \) to show that \( \text{cl}(A) = M \) and \( \text{cl}(B) = N \). Then it follows that \( \text{aff}(A) = \text{aff}(B) = \mathbb{R}^n \).

(b) For \( A \cap B \neq \emptyset \), choose \( A = \mathbb{R}^n \) and \( B \) a hyperplane. For \( A \) and \( B \) not convex, choose \( A = \mathbb{R}^n \setminus H \) and \( B = H \), where \( H \) is a hyperplane.

4.13 Follow the proof of Theorem 4.12.