

Math 341 Homework 7 (due Mar. 20)

Problems from the book: (pp. 23-24) 4.4, 4.8, 4.9, 4.10, 4.13

**Hints**

4.4 Use Caratheodory theorem and Theorem 4.13. We want to show that  $\text{conv}(A) \cap \text{conv}(B) = \emptyset$  if any  $n + 1$  points of  $B$  can be strictly separated from  $A$  by a hyperplane. If  $\text{conv}(A) \cap \text{conv}(B) \neq \emptyset$ , then there is a point  $x \in \text{conv}(A) \cap \text{conv}(B)$ . By Caratheodory theorem,  $x \in \text{conv}\{x_1, x_2, \dots, x_{n+1}\}$  with  $T = \{x_1, x_2, \dots, x_{n+1}\} \subset B$ . Since there is a hyperplane strictly separating  $T$  and  $A$ ,  $\text{conv}(T) \cap \text{conv}(A) = \emptyset$ . Contradiction.

4.8 Use Theorem 4.7. Since  $A \cap \text{int}(B) = \emptyset$  and  $\text{int}(B) = \emptyset$ ,  $A$  and  $B$  can be separated by a hyperplane  $H = \{f(x) = \alpha\}$ . Suppose that  $A \subset M = \{f(x) \geq \alpha\}$  and  $B \subset N = \{f(x) \leq \alpha\}$ . So  $\text{bd}(B) \subset N$ . Show that  $A \subset H$ .

4.9 You need to show that  $\text{cl}(S) \neq \mathbb{R}^n$ . If  $\dim(S) < n$ , then  $\text{cl}(S) \subset \text{aff}(S) \neq \mathbb{R}^n$ . If  $\dim(S) = n$ , then  $\text{int}(S) \neq \emptyset$ ; pick a point  $p \notin S$  and apply Theorem 4.7 to  $\{p\}$  and  $S$ . Show that  $\text{cl}(S)$  is contained in a closed half-space.

4.10 (a) Without the loss of generality, you may assume that  $A \cup B = \mathbb{R}^n$ . Show that at least one of  $A$  and  $B$  has nonempty interior. By Theorem 4.7,  $A$  and  $B$  are separated by a hyperplane  $H = \{f(x) = \alpha\}$ . Suppose that  $A \subset M = \{f(x) \geq \alpha\}$  and  $B \subset N = \{f(x) \leq \alpha\}$ . Use the fact that  $A \cup B = \mathbb{R}^n$  to show that  $\text{cl}(A) = M$  and  $\text{cl}(B) = N$ . Then it follows that  $\text{aff}(A) = \text{aff}(B) = \mathbb{R}^n$ .

(b) For  $A \cap B \neq \emptyset$ , choose  $A = \mathbb{R}^n$  and  $B$  a hyperplane. For  $A$  and  $B$  not convex, choose  $A = \mathbb{R}^n \setminus H$  and  $B = H$ , where  $H$  is a hyperplane.

4.13 Follow the proof of Theorem 4.12.