Math 341 Homework 6 Solution

3.8 (p. 33) Let $u = (a_1, a_2, a_3, a_4)$ and $H = \{\langle u, x\rangle = \alpha\}$. So we are supposed to solve the system of linear equations

$$\begin{cases} a_1 & + & & & a_3 & & & - & \alpha & = & 0 \\ 2a_1 & + & 3a_2 & + & a_3 & & & - & \alpha & = & 0 \\ a_1 & + & 2a_2 & + & 2a_3 & & & - & \alpha & = & 0 \\ a_1 & + & a_2 & + & a_3 & + & a_4 & - & \alpha & = & 0 \end{cases}$$

with unknown $(a_1, a_2, a_3, a_4, \alpha)$. Use Guassian reduction:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & -1 & 0 \\ 2 & 3 & 1 & 0 & -1 & 0 \\ 1 & 2 & 2 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -3 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & -3 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 0 & -5 & 1 & 0 \end{bmatrix}$$

So $(3, -1, 2, 1, 5)$ is a solution. Therefore, $u = (3, -1, 2, 1, 5)$ and $H = \{3x_1 - x_2 + 2x_3 + x_4 = 5\}$. Of course, any multiples of $(3, -1, 2, 1, 5)$ are valid answers.

3.10 (p. 33) (a) Use 3.9. Since $\text{aff}(S)$ is closed, $\text{cl}(\text{aff}(S)) = \text{aff}(S)$. Since $S \subset \text{aff}(S)$, $\text{cl}(S) \subset \text{cl}(\text{aff}(S))$.

(b) Since $S \subset \text{cl}(S)$, $\text{aff}(S) \subset \text{aff}(\text{cl}(S))$. By (a), $\text{cl}(S) \subset \text{aff}(S) \Rightarrow \text{aff}(\text{cl}(S)) \subset \text{aff}(\text{aff}(S)) = \text{aff}(S)$. Therefore, $\text{aff}(S) = \text{aff}(\text{cl}(S))$.

(c) Use 2.20. Let $y \in \text{aff}(S)$ and $x \in \text{relint}(S)$. Then $\text{relint}\,\overline{xy} \cap \text{relint}(S) \neq \emptyset$ by 2.20. Let $z \in \text{relint}\,\overline{xy} \cap \text{relint}(S)$. Then y is an affine combination of z and x. Hence $y \in \text{aff}(\text{relint}(S))$. Therefore, $\text{aff}(S) \subset \text{aff}(\text{relint}(S))$.

On the other hand, $\text{relint}(S) \subset S \Rightarrow \text{aff}(\text{relint}(S)) \subset \text{aff}(S)$. Therefore, $\text{aff}(\text{relint}(S)) = \text{aff}(S)$.

3.11 (p. 33) Two hyperplanes H_1 and H_2 are parallel to each other if $H_2 = x_0 + H_1$ for some x_0.

Suppose that H_1 and H_2 are parallel to each other such that $H_2 = x_0 + H_1$. Let $H_1 = \{\langle u, x\rangle = \alpha\}$ and $\beta = \langle u, x_0\rangle + \alpha$. We will show that $H_2 = \{\langle u, x\rangle = \beta\}$.

Let $x \in H_2$. Then $x = x_0 + y$ for some $y \in H_1$. Then

$$\langle u, x\rangle = \langle u, x_0 + y\rangle = \langle u, x_0\rangle + \langle u, y\rangle = \langle u, x_0\rangle + \alpha = \beta$$

Therefore, $x \in \{\langle u, x\rangle = \beta\}$ and $H_2 \subset \{\langle u, x\rangle = \beta\}$.

Let $x \in \{\langle u, x\rangle = \beta\}$. Then

$$\langle u, x - x_0\rangle = \langle u, x\rangle - \langle u, x_0\rangle = \beta - \langle u, x_0\rangle = \alpha$$

Therefore, $x - x_0 \in H_1$ and $x \in x_0 + H_1 = H_2$. So $\{\langle u, x\rangle = \beta\} \subset H_2$ and $\{\langle u, x\rangle = \beta\} = H_2$. Consequently, u is the normal vector of both H_1 and H_2.

On the other hand, assume that the normal vectors of H_1 and H_2 are multiples of each other. Let $H_1 = \{\langle u, x\rangle = \alpha\}$ and $H_2 = \{\langle v, x\rangle = \beta\}$ with $v = \lambda u$ for some $\lambda \neq 0$.

Let $x_1 \in H_1$ and $x_2 \in H_2$. We will show that $H_1 = (x_1 - x_2) + H_2$.

Let $x \in H_1$. Then

$$\begin{aligned}\langle v, x + x_2 - x_1\rangle &= \langle v, x\rangle + \langle v, x_2\rangle - \langle v, x_1\rangle = \langle v, x\rangle - \langle v, x_1\rangle + \beta \\ &= \langle \lambda u, x\rangle - \langle \lambda u, x_1\rangle + \beta = \lambda\langle u, x\rangle - \lambda\langle u, x_1\rangle + \beta \\ &= \lambda\alpha - \lambda\alpha + \beta = \beta\end{aligned}$$

So $x + x_2 - x_1 \in H_2$ and $x \in (x_1 - x_2) + H_2$. Therefore, $H_1 \subset (x_1 - x_2) + H_2$. Similarly, $H_2 \subset (x_2 - x_1) + H_1$, i.e., $(x_1 - x_2) + H_2 \subset H_1$. So $H_1 = (x_1 - x_2) + H_2$ and H_1 and H_2 are parallel.

3.14 (p. 33) (a) Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$ be two points in $f(A)$, where $x_1, x_2 \in A$. Then $\alpha y_1 + \beta y_2 = \alpha f(x_1) + \beta f(x_2) = f(\alpha x_1 + \beta x_2)$ for $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$. Since A is convex, $\alpha x_1 + \beta x_2 \in A$ and hence $f(\alpha x_1 + \beta x_2) \in f(A)$. So $\alpha y_1 + \beta y_2 \in f(A)$ and $f(A)$ is convex.

(b) Let $x_1, x_2 \in f^{-1}(B)$. Then $f(x_1), f(x_2) \in B$. Since B is convex, $f(\alpha x_1 + \beta x_2) = \alpha f(x_1) + \beta f(x_2) \in B$ for $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$. Therefore, $\alpha x_1 + \beta x_2 \in f^{-1}(B)$ and $f^{-1}(B)$ is convex.

4.1 (p. 40) By Theorem 4.12, $\mathrm{conv}(B)$ cannot be closed. So we are looking for a closed set B such that $\mathrm{conv}(B)$ is not closed. Take $B = \{y = x^2, x \geq 0\}$. Then $\mathrm{conv}(B) = \{y \geq x^2, x > 0\} \cup \{(0, 0)\}$. Let $A = \{x = 0, 1 \leq y \leq 2\}$.

Since A is convex, $\mathrm{conv}(A) = A$. Obviously, $\mathrm{conv}(A) \cap \mathrm{conv}(B) = \emptyset$.

Next, we show that A and B cannot be strictly separated by a line. Suppose that A and B is strictly separated by a line L. Since L strictly separate A and B, $L \cap A = L \cap B = \emptyset$.

If the slope of L is infinite, then $L = \{x = c\}$. If $c \leq 0$, then $A, B \subset \{x \geq c\}$ and hence L does not separate A and B; otherwise, if $c > 0$, $L \cap B \neq \emptyset$. Either way we have a contradiction.

Suppose that L has finite slope. Let $L = \{y - kx = b\}$. We have either $B \subset \{y - kx < b\}$ or $B \subset \{y - kx > b\}$.

If $B \subset \{y - kx < b\}$, then $x^2 - kx < b$ for all $x \geq 0$. This is impossible since $\lim_{x\to\infty}(x^2 - kx) = \infty$. Therefore, $B \subset \{y - kx > b\}$ and $A \subset \{y - kx < b\}$.

Since $B \subset \{y - kx > b\}$, $(0, 0) \in \{y - kx > b\}$ and hence $0 > b$. On the other hand, since $A \subset \{y - kx < b\}$, $(0, 1) \in \{y - kx < b\}$ and hence $1 < b$. Contradiction.

4.2 (p. 40) Let $\{F_\lambda : \lambda \in I\}$ be the collection of all closed half-spaces that contain S. Obviously, $S \subset \cap_{\lambda\in I} F_\lambda$. We want to show that $S \supset \cap_{\lambda\in I} F_\lambda$.

Let $p \in \cap_{\lambda\in I} F_\lambda$. Suppose that $p \notin S$.

Since $\{p\}$ and S are convex, $\{p\}$ is compact and S is closed, by Theorem 4.12, $\{p\}$ and S are strictly separated by a hyperplane $H = \{f(x) = \alpha\}$. Suppose that $p \in M = \{f(x) \geq \alpha\}$ and $S \subset N = \{f(x) \leq \alpha\}$. Since H strictly separates $\{p\}$ and S, $p \notin H$ and hence $p \notin N$. Since $S \subset N$, $N \in \{F_\lambda : \lambda \in I\}$ and hence $p \in \cap_{\lambda \in I} F_\lambda \subset N$. Contradiction.

Therefore, $p \in S$ and $\cap_{\lambda \in I} F_\lambda \subset S$. So $\cap_{\lambda \in I} F_\lambda = S$.

4.3 (p. 40) Since π is linear and S is convex, $\pi(S)$ is convex by 3.14. To show that $\pi(S)$ is relative open in G, we prove first the fact that $\pi(B(x,\delta)) = B(\pi(x),\delta) \cap G$.

Let $x = (x_1, x_2, ..., x_n)$. For every $y = (y_1, y_2, ..., y_n) \in B(x, \delta)$, we have

$$(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_n - y_n)^2 < \delta^2$$

Then

$$\begin{aligned} ||\pi(y) - \pi(x)||^2 &= (x_{k+1} - y_{k+1})^2 + ... + (x_n - y_n)^2 \\ &\leq (x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_n - y_n)^2 < \delta^2 \end{aligned}$$

So $\pi(y) \in B(\pi(x), \delta)$ and $\pi(B(x,\delta)) \subset B(\pi(x),\delta) \cap G$.

On the other hand, for each $y = (0, 0, ..., 0, y_{k+1}, ..., y_n) \in B(\pi(x), \delta) \cap G$, $y = \pi(w)$, where $w = (x_1, x_2, ..., x_k, y_{k+1}, ..., y_n)$. Since $||w - x|| = ||y - \pi(x)|| < \delta$, $w \in B(x, \delta)$ so $y \in \pi(B(x, \delta))$. Therefore, $B(\pi(x), \delta) \cap G \subset \pi(B(x, \delta))$ and $\pi(B(x,\delta)) = B(\pi(x),\delta) \cap G$.

Since S is open, for every point $x \in S$, there exists an open ball $B(x, \delta) \subset S$. Since $\pi(B(x,\delta)) = B(\pi(x),\delta) \cap G$, $\pi(S)$ is open in G.

4.5 (p. 40) Let $S = \{x \in \mathbb{R}^n : f(x) \geq c\}$ be a half-space in $\mathbb{R}^n$, where $f(x)$ is a linear functional $f : \mathbb{R}^n \to \mathbb{R}$.

Let $B = [c, \infty) \subset \mathbb{R}$. Obviously, $S = f^{-1}(B)$. Since B is convex, $S = f^{-1}(B)$ is convex by 3.14.

4.6 (p. 40) By Theorem 4.7, A and B are separated by a hyperplane H. We will show that H actually strictly separates A and B. Let $H = \{f(x) = c\}$. Suppose that $A \subset U = \{f(x) \geq c\}$ and $B \subset V = \{f(x) \leq c\}$.

Since A is open, $A \subset \text{int}(U) = \{f(x) > c\}$. Similarly, $B \subset \text{int}(V) = \{f(x) < c\}$. Therefore, H strictly separates A and B.