3.8 (p. 33) Let \( u = (a_1, a_2, a_3, a_4) \) and \( H = \{ \langle u, x \rangle = \alpha \} \). So we are supposed to solve the system of linear equations
\[
\begin{cases}
    a_1 + a_3 - \alpha = 0 \\
    2a_1 + 3a_2 + a_3 - \alpha = 0 \\
    a_1 + 2a_2 + 2a_3 - \alpha = 0 \\
    a_1 + a_2 + a_3 + a_4 - \alpha = 0
\end{cases}
\]
with unknown \((a_1, a_2, a_3, a_4, \alpha)\). Use Guassian reduction:
\[
\begin{bmatrix}
    1 & 0 & 1 & 0 & -1 & 0 \\
    2 & 3 & 1 & 0 & -1 & 0 \\
    1 & 2 & 2 & 0 & -1 & 0 \\
    1 & 1 & 1 & 1 & -1 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
    1 & 0 & 1 & 0 & -1 & 0 \\
    0 & 1 & 0 & 1 & 0 & 0 \\
    0 & 3 & -1 & 0 & 1 & 0 \\
    0 & 2 & 1 & 0 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
    1 & 0 & 0 & -3 & 0 & 0 \\
    0 & 1 & 0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 3 & -1 & 0 \\
    0 & 0 & 0 & -5 & 1 & 0
\end{bmatrix}
\]
So \((3, -1, 2, 1, 5)\) is a solution. Therefore, \( u = (3, -1, 2, 1, 5) \) and \( H = \{ 3x_1 - x_2 + 2x_3 + x_4 = 5 \} \). Of course, any multiples of \((3, -1, 2, 1, 5)\) are valid answers.

3.10 (p. 33) (a) Use 3.9. Since \( \text{aff}(S) \) is closed, \( \text{cl} (\text{aff}(S)) = \text{aff}(S) \). Since \( S \subset \text{aff}(S) \), \( \text{cl}(S) \subset \text{cl}(\text{aff}(S)) \).

(b) Since \( S \subset \text{cl}(S) \), \( \text{aff}(S) \subset \text{aff}(\text{cl}(S)) \). By (a), \( \text{cl}(S) \subset \text{aff}(\text{cl}(S)) \Rightarrow \text{aff}(\text{cl}(S)) \subset \text{aff}(\text{aff}(S)) = \text{aff}(S) \). Therefore, \( \text{aff}(S) = \text{aff}(\text{cl}(S)) \).

(c) Use 2.20. Let \( y \in \text{aff}(S) \) and \( x \in \text{relint}(S) \). Then \( \text{relint} \mathcal{F} \cap \text{relint}(S) \neq \emptyset \) by 2.20. Let \( z \in \text{relint} \mathcal{F} \cap \text{relint}(S) \). Then \( y \) is an affine combination of \( z \) and \( x \). Hence \( y \in \text{aff} (\text{relint}(S)) \). Therefore, \( \text{aff}(S) \subset \text{aff}(\text{relint}(S)) \).

On the other hand, \( \text{relint}(S) \subset S \Rightarrow \text{aff}(\text{relint}(S)) \subset \text{aff}(S) \). Therefore, \( \text{aff}(\text{relint}(S)) = \text{aff}(S) \).

3.11 (p. 33) Two hyperplanes \( H_1 \) and \( H_2 \) are parallel to each other if \( H_2 = x_0 + H_1 \) for some \( x_0 \).

Suppose that \( H_1 \) and \( H_2 \) are parallel to each other such that \( H_2 = x_0 + H_1 \). Let \( H_1 = \{ \langle u, x \rangle = \alpha \} \) and \( \beta = \langle u, x_0 \rangle + \alpha \). We will show that \( H_2 = \{ \langle u, x \rangle = \beta \} \).

Let \( x \in H_2 \). Then \( x = x_0 + y \) for some \( y \in H_1 \). Then
\[
\langle u, x \rangle = \langle u, x_0 + y \rangle = \langle u, x_0 \rangle + \langle u, y \rangle = \langle u, x_0 \rangle + \alpha = \beta
\]
Therefore, \( x \in \{ \langle u, x \rangle = \beta \} \) and \( H_2 \subset \{ \langle u, x \rangle = \beta \} \).
Let \( x \in \{ \langle u, x \rangle = \beta \} \). Then
\[
\langle u, x - x_0 \rangle = \langle u, x \rangle - \langle u, x_0 \rangle = \beta - \langle u, x_0 \rangle = \alpha
\]
Therefore, \( x - x_0 \in H_1 \) and \( x \in x_0 + H_1 = H_2 \). So \( \{ \langle u, x \rangle = \beta \} \subset H_2 \) and \( \{ \langle u, x \rangle = \beta \} = H_2 \). Consequently, \( u \) is the normal vector of both \( H_1 \) and \( H_2 \).

On the other hand, assume that the normal vectors of \( H_1 \) and \( H_2 \) are multiples of each other. Let \( H_1 = \{ \langle u, x \rangle = \alpha \} \) and \( H_2 = \{ \langle v, x \rangle = \beta \} \) with \( v = \lambda u \) for some \( \lambda \neq 0 \).

Let \( x_1 \in H_1 \) and \( x_2 \in H_2 \). We will show that \( H_1 = (x_1 - x_2) + H_2 \).

Let \( x \in H_1 \). Then
\[
\langle v, x + x_2 - x_1 \rangle = \langle v, x \rangle + \langle v, x_2 \rangle - \langle v, x_1 \rangle = \langle v, x \rangle - \langle v, x_1 \rangle + \beta
\]
\[
= (\lambda u, x) - (\lambda u, x_1) + \beta = \lambda \langle u, x \rangle - \lambda \langle u, x_1 \rangle + \beta
\]
\[
= \lambda \alpha - \lambda \alpha + \beta = \beta
\]
So \( x + x_2 - x_1 \in H_2 \) and \( x \in (x_1 - x_2) + H_2 \). Therefore, \( H_1 \subset (x_1 - x_2) + H_2 \). Similarly, \( H_2 \subset (x_2 - x_1) + H_1 \), i.e., \( (x_1 - x_2) + H_2 \subset H_1 \). So \( H_1 = (x_1 - x_2) + H_2 \) and \( H_1 \) and \( H_2 \) are parallel.

**3.14 (p. 33)** (a) Let \( y_1 = f(x_1) \) and \( y_2 = f(x_2) \) be two points in \( f(A) \), where \( x_1, x_2 \in A \). Then \( \alpha y_1 + \beta y_2 = \alpha f(x_1) + \beta f(x_2) = f(\alpha x_1 + \beta x_2) \) for \( \alpha, \beta \geq 0 \) and \( \alpha + \beta = 1 \). Since \( A \) is convex, \( \alpha x_1 + \beta x_2 \in A \) and hence \( f(\alpha x_1 + \beta x_2) \in f(A) \). So \( \alpha y_1 + \beta y_2 \in f(A) \) and \( f(A) \) is convex.

(b) Let \( x_1, x_2 \in f^{-1}(B) \). Then \( f(x_1), f(x_2) \in B \). Since \( B \) is convex, \( f(\alpha x_1 + \beta x_2) = \alpha f(x_1) + \beta f(x_2) \in B \) for \( \alpha, \beta \geq 0 \) and \( \alpha + \beta = 1 \). Therefore, \( f^{-1}(B) \) is convex.

**4.1 (p. 40)** By Theorem 4.12, \( \text{conv}(B) \) cannot be closed. So we are looking for a closed set \( B \) such that \( \text{conv}(B) \) is not closed. Take \( B = \{ y = x^2, x \geq 0 \} \). Then \( \text{conv}(B) = \{ y \geq x^2, x > 0 \} \cup \{ (0, 0) \} \). Let \( A = \{ x = 0, 1 \leq y \leq 2 \} \).

Since \( A \) is convex, \( \text{conv}(A) = A \). Obviously, \( \text{conv}(A) \cap \text{conv}(B) = \emptyset \).

Next, we show that \( A \) and \( B \) cannot be strictly separated by a line. Suppose that \( A \) and \( B \) is strictly separated by a line \( L \). Since \( L \) strictly separate \( A \) and \( B \), \( L \cap A = L \cap B = \emptyset \).

If the slope of \( L \) is infinite, then \( L = \{ x = c \} \). If \( c \leq 0 \), then \( A, B \subset \{ x \geq c \} \) and hence \( L \) does not separate \( A \) and \( B \); otherwise, if \( c > 0 \), \( L \cap B = \emptyset \). Either way we have a contradiction.

Suppose that \( L \) has finite slope. Let \( L = \{ y - kx = b \} \). We have either \( B \subset \{ y - kx < b \} \) or \( B \subset \{ y - kx > b \} \).

If \( B \subset \{ y - kx < b \} \), then \( x^2 - kx < b \) for all \( x \geq 0 \). This is impossible since \( \lim_{x \to \infty}(x^2 - kx) = \infty \). Therefore, \( B \subset \{ y - kx > b \} \) and \( A \subset \{ y - kx < b \} \).

Since \( B \subset \{ y - kx > b \} \), \( (0, 0) \in \{ y - kx > b \} \) and hence \( 0 > b \). On the other hand, since \( A \subset \{ y - kx < b \} \), \( (0, 1) \in \{ y - kx < b \} \) and hence \( 1 < b \). Contradiction.

**4.2 (p. 40)** Let \( \{ F_\lambda : \lambda \in I \} \) be the collection of all closed half-spaces that contain \( S \). Obviously, \( S \subset \cap_{\lambda \in I} F_\lambda \). We want to show that \( S \supset \cap_{\lambda \in I} F_\lambda \).

Let \( p \in \cap_{\lambda \in I} F_\lambda \). Suppose that \( p \notin S \).
Since \( \{p\} \) and \( S \) are convex, \( \{p\} \) is compact and \( S \) is closed, by Theorem 4.12, \( \{p\} \) and \( S \) are strictly separated by a hyperplane \( H = \{f(x) = \alpha\} \). Suppose that \( p \in M = \{f(x) \geq \alpha\} \) and \( S \subset N = \{f(x) \leq \alpha\} \). Since \( H \) strictly separates \( \{p\} \) and \( S \), \( p \notin H \) and hence \( p \notin N \). Since \( S \subset N \), \( N \in \{F_\lambda : \lambda \in I\} \) and hence \( p \in \cap_{\lambda \in I} F_\lambda \subset N \). Contradiction.

Therefore, \( p \in S \) and \( \cap_{\lambda \in I} F_\lambda \subset S \). So \( \cap_{\lambda \in I} F_\lambda = S \).

4.3 (p. 40) Since \( \pi \) is linear and \( S \) is convex, \( \pi(S) \) is convex by 3.14. To show that \( \pi(S) \) is relative open in \( G \), we prove first the fact that \( \pi(B(x, \delta)) = B(\pi(x), \delta) \cap G \).

Let \( x = (x_1, x_2, \ldots, x_n) \). For every \( y = (y_1, y_2, \ldots, y_n) \in B(x, \delta) \), we have
\[
(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2 < \delta^2
\]
Then
\[
\|\pi(y) - \pi(x)\|^2 = (x_{k+1} - y_{k+1})^2 + \ldots + (x_n - y_n)^2
\]
\[
\leq (x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2 < \delta^2
\]
So \( \pi(y) \in B(\pi(x), \delta) \) and \( \pi(B(x, \delta)) \subset B(\pi(x), \delta) \cap G \).

On the other hand, for each \( y = (0, 0, \ldots, 0, y_{k+1}, \ldots, y_n) \in B(\pi(x), \delta) \cap G \), \( y = \pi(w) \), where \( w = (x_1, x_2, \ldots, x_k, y_{k+1}, \ldots, y_n) \). Since \( ||w - x|| = ||y - \pi(x)|| < \delta \), \( w \in B(x, \delta) \) so \( y \in \pi(B(x, \delta)) \). Therefore, \( B(\pi(x), \delta) \cap G \subset \pi(B(x, \delta)) \) and \( \pi(B(x, \delta)) = B(\pi(x), \delta) \cap G \).

Since \( S \) is open, for every point \( x \in S \), there exists an open ball \( B(x, \delta) \subset S \). Since \( \pi(B(x, \delta)) = B(\pi(x), \delta) \cap G \), \( \pi(S) \) is open in \( G \).

4.5 (p. 40) Let \( S = \{x \in \mathbb{R}^n : f(x) \geq c\} \) be a half-space in \( \mathbb{R}^n \), where \( f(x) \) is a linear functional \( f : \mathbb{R}^n \to \mathbb{R} \).

Let \( B = [c, \infty) \subset \mathbb{R} \). Obviously, \( S = f^{-1}(B) \). Since \( B \) is convex, \( S = f^{-1}(B) \) is convex by 3.14.

4.6 (p. 40) By Theorem 4.7, \( A \) and \( B \) are separated by a hyperplane \( H \). We will show that \( H \) actually strictly separates \( A \) and \( B \). Let \( H = \{f(x) = c\} \). Suppose that \( A \subset U = \{f(x) \geq c\} \) and \( B \subset V = \{f(x) \leq c\} \).

Since \( A \) is open, \( A \subset \text{int}(U) = \{f(x) > c\} \). Similarly, \( B \subset \text{int}(V) = \{f(x) < c\} \). Therefore, \( H \) strictly separates \( A \) and \( B \).