Math 341 Homework 5 Solution

2.37 (p. 25) (a) Let $B = \{x^2 + y^2 \leq 1\}$ be a disk and p be a point on the circle $\text{bd}(B) = \{x^2 + y^2 = 1\}$. Let $S = B \backslash \{p\}$. First, we show that S is convex. Let $u, v \in S$ and $w \in \text{relint}(\overline{xy})$. Since $u, v \in B$ and B is convex, $w \in B$. Obviously, $w \notin \text{bd}(B)$ and hence $w \neq p$. Therefore, $w \in S$ and $\overline{uv} \subset S$. So S is convex. Next, we show that S is enclosed by B. Let A be the set with $S \subset A \subset B$. Then $A \backslash S \subset B \backslash S = \{p\}$. So $A \backslash S = \emptyset$ or $\{p\}$. That is, $A = S$ or B. Therefore, there are no sets between S and B and S is enclosed by B. Finally, we need to show that S is enclosed by exactly one convex set. Suppose that S is enclosed by a convex set A. Let q be a point in A such that $q \notin S$. If $p = q$, then $S \subset B \subset A$ and hence $A = B$. Suppose that $p \neq q$. Then $q \notin B$. So $d(o, q) > 1$, where o is the origin. Since A is convex, $\overline{oq} \subset A$, i.e., $\lambda q \in A$ for every $1 \geq \lambda \geq 0$. Let $r = d(o, q)$. Then

$$x = \frac{1 + r}{2r} q \in A$$

but $x \notin B$ because $d(o, x) = ||x|| > 1$. Then $S \subset \text{conv}(S \cup \{x\}) \subset A$ with $S \neq \text{conv}(S \cup \{x\}) \neq A$. Contradiction.

Similarly, we can construct convex sets enclosed by exactly n convex sets for all n. Let $G = \{p_1, p_2, ..., p_n, ...\} \subset \text{bd}(B)$, $A_n = B \backslash \{p_1, p_2, ..., p_n\}$ and $A = B \backslash G$.

Then $A, A_1, A_2, ..., A_n, ...$ are convex; A_n is enclosed by exactly n convex sets, which are $A_n \cup \{p_1\}$, $A_n \cup \{p_2\}$, ..., and $A_n \cup \{p_n\}$; and A is enclosed by infinitely many convex sets, which are $A \cup \{p_1\}, A \cup \{p_2\}, ..., A \cup \{p_n\}, ...,$.

(b) Let $D = \{x^2 + y^2 < 1\}$ and $G = \{p_1, p_2, ..., p_n, ...\} \subset \text{bd}(D)$. Let $A_i = D \cup \{p_1, p_2, ..., p_i\}$. Then $A_1 \subset A_2 \subset ... \subset A_i \subset A_{i+1} \subset ...$ with A_{i+1} enclosing A_i.

(c) This is true. Suppose that there exist two distinct points $p, q \in A$ and $p, q \notin B$. Then $B \subset \text{conv}(B \cup \{p\}) \subset A$. Since A encloses B, there are no convex sets between B and A and hence $\text{conv}(B \cup \{p\}) = A$. Therefore, $q \in \text{conv}(B \cup \{p\}) \Rightarrow q$ is a convex combination of $x_1, x_2, ..., x_{n-1}, p$ with $x_1, x_2, ..., x_{n-1} \in B$. So there exist $\lambda_1, \lambda_2, ..., \lambda_{n-1}, \lambda_n \geq 0$ such that $\sum_{i=1}^n \lambda_i = 1$ and

$$q = \lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_{n-1} x_{n-1} + \lambda_n p.$$

We let

$$y = \frac{1}{\lambda_1 + \lambda_2 + ... + \lambda_{n-1}} (\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_{n-1} x_{n-1})$$

Then y is a convex combination of $x_1, x_2, ..., x_{n-1}$ and hence $y \in B$. And

$$q = (\lambda_1 + \lambda_2 + ... + \lambda_{n-1}) y + \lambda_n p = (1 - \lambda_n) y + \lambda_n p$$

and hence $q \in \overline{py}$ and $q \neq p, y$.

By the same argument, we can show that $p \in \overline{xq}$ for some $x \in B$. Therefore,

$$q = \alpha_1 p + \beta_1 y$$
$$p = \alpha_2 q + \beta_2 x$$

where $\alpha_i, \beta_i > 0$ and $\alpha_i + \beta_i = 1$. We may eliminate q by multiplying the first equation by α_2 and adding it to the second equation:

$$p = \alpha_1 \alpha_2 p + \alpha_2 \beta_1 y + \beta_2 x$$

So

$$p = \frac{1}{1 - \alpha_1 \alpha_2}(\beta_2 x + \alpha_2 \beta_1 y)$$

Since

$$\begin{aligned} \beta_2 + \alpha_2 \beta_1 &= 1 - \alpha_2 + \alpha_2 \beta_1 \\ &= 1 - (1 - \beta_1)\alpha_2 = 1 - \alpha_1 \alpha_2 \end{aligned}$$

p is a convex combination of x and y, i.e., $p \in \overline{xy}$. Then $p \in B$, which is a contradiction.

(d) This is false. Let x_1, x_2, x_3 be three points on $\mathbb{R}^2$ not lying on a line and $\Delta x_1 x_2 x_3$ be the triangle with vertices at x_1, x_2, x_3 (including the boundary). Let $B = (\Delta x_1 x_2 x_3 \backslash \overline{x_1 x_2}) \cup \{x_1\}$ (we remove the edge $x_1 x_2$ from the triangle but keeping one vertex x_1. We will show that B is convex but $A = B \cup \{x_2\}$ is not.

To show that B is convex, let $u, v \in B$ and we show that $\overline{uv} \subset B$. Since $u, v \in \Delta x_1 x_2 x_3$,

$$u = a_1 x_1 + a_2 x_2 + a_3 x_3$$
$$v = b_1 x_1 + b_2 x_2 + b_3 x_3$$

where $a_i, b_i \geq 0$ and $a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = 1$. Let $w = \alpha u + \beta v \in \text{relint}(\overline{uv})$, where $\alpha, \beta > 0$ and $\alpha + \beta = 1$. Then

$$\alpha u + \beta v = (\alpha a_1 + \beta b_1) x_1 + (\alpha a_2 + \beta b_2) x_2 + (\alpha a_3 + \beta b_3) x_3$$

If either $a_3 > 0$ or $b_3 > 0$, then $\alpha a_3 + \beta b_3 > 0$ and hence $w \notin \overline{x_1 x_2}$; and since $w \in \Delta x_1 x_2 x_3$, $w \in B$. Suppose that $a_3 = b_3 = 0$. Then $u, v \in \overline{x_1 x_2}$. And since $u, v \in B$, $u, v = x_1$. Then $w = x_1 \in B$. So B is convex.

Since $x_1, x_2 \in A$ and $\overline{x_1 x_2} \not\subset A$, A is not convex.

3.1 (p. 32) By the definition of normal vector, $H = \{x : \langle u, x \rangle = \alpha\}$. Since $x_1, x_2 \in H$, $\langle u, x_1 \rangle = \langle u, x_2 \rangle = \alpha$. So $\langle u, x_1 \rangle - \langle u, x_2 \rangle = 0$ and $\langle u, x_1 - x_2 \rangle = 0$, i.e., u is orthogonal to $x_1 - x_2$.

3.2 (p. 32) Let $e_1, e_2, ..., e_{n-1}$ be a basis for V.

Since $y \notin V$, $\{e_1, e_2, ..., e_{n-1}, y\}$ is linearly independent and hence a basis for $\mathbb{R}^n$. Therefore every point $p \in \mathbb{R}$ is a linear combination of $e_1, e_2, ..., e_{n-1}, y$, i.e.,

$$p = a_1 e_1 + a_2 e_2 + ... + a_{n-1} e_{n-1} + \alpha y$$

Let $x = a_1 e_1 + a_2 e_2 + ... + a_{n-1} e_{n-1}$. Then $x \in V$ and $p = x + \alpha y$.

Next we show such representation is unique. Suppose that $p = x + \alpha y = x' + \alpha' y$ with $x, x' \in V$. Then $x - x' = (\alpha' - \alpha)y$. If $x = x'$, $\alpha = \alpha'$. If $x \neq x'$, $\alpha' \neq \alpha$; then $y = (\alpha' - \alpha)^{-1}(x - x') \in V$. Contradiction.

3.3 (p. 32) We want to show that $x_0 + H_1 = [f : \beta]$.

Let $x \in x_0 + H_1$. Then $x = x_0 + y$ with $y \in H_1$. Since $y \in H_1$, $f(y) = \alpha$. So $f(x) = f(x_0) + f(y) = f(x_0) + \alpha = \beta$. Therefore, $x \in [f : \beta]$ and $x_0 + H_1 \subset [f : \beta]$.

Let $x \in [f : \beta]$. Then $f(x) = \beta$. So $f(x - x_0) = f(x) - f(x_0) = \beta - f(x_0) = \alpha$. Hence $x - x_0 \in H_1$ and $x = x_0 + (x - x_0) \in x_0 + H_1$. So $[f : \beta] \subset x_0 + H_1$.

In conclusion, $x_0 + H_1 = [f : \beta]$.

3.4 (p. 32) (a) $u = (2, -3)$

(b) $(-1, 1) + H = \{2x - 3y = -3\}$.

3.5 (p. 32) We show that $F_1 = F_2$ if $x_1 - x_2 \in V$ and $F_1 \cap F_2 = \emptyset$.

Suppose that $x_1 - x_2 \in V$. Then $(x_1 - x_2) + V = V \Rightarrow x_1 + V = x_2 + V$, i.e., $F_1 = F_2$.

Suppose that $x_1 - x_2 \notin V$. We want to show that $F_1 \cap F_2 = \emptyset$. Otherwise, assume that $F_1 \cap F_2 \neq \emptyset$. Let $x \in F_1 \cap F_2$. Since $x \in F_1 = x_1 + V$, $x - x_1 \in V$; since $x \in F_2 = x_2 + V$, $x - x_2 \in V$. And since V is a linear subspace, $(x - x_2) - (x - x_1) \in V$, i.e., $x_1 - x_2 \in V$. Contradiction. So $F_1 \cap F_2 = \emptyset$.

3.6 (p. 33) $L = \{x + 4y = 7\}$.

3.9 (p. 33) Let $F = W + x$ where $x \in \mathbb{R}^n$ and W is a linear subspace of $\mathbb{R}^n$. There is nothing to prove if $F = \mathbb{R}^n$. Let us assume that $F \neq \mathbb{R}^n$. So W is a proper subspace of $\mathbb{R}^n$.

Choose an orthonormal basis $e_1, e_2, ..., e_n$ for $\mathbb{R}^n$ with $e_1, e_2, ..., e_k$ generating W and $||e_i|| = 1$ for $i = 1, 2, ..., n$.

Let $y \notin F$ and $y - x = a_1 e_1 + a_2 e_2 + ... + a_k e_k + a_{k+1} e_{k+1} + a_{k+2} e_{k+2} + ... + a_n e_n$. Since $y - x \notin W$, $a_{k+1}, a_{k+1}, ..., a_n$ are not all zero. Let

$$r = \sqrt{a_{k+1}^2 + a_{k+2}^2 + ... + a_n^2}.$$

Then $r > 0$.

Let $z \in F$ and $z = x + b_1 e_1 + b_2 e_2 + ... + b_k e_k$. Then

$$\begin{aligned} ||y - z|| &= ||(a_1 - b_1)e_1 + (a_2 - b_2)e_2 + ... + (a_k - b_k)e_k \\ &\quad + a_{k+1}e_{k+1} + a_{k+2}e_{k+2} + ... + a_n e_n|| \\ &= \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + ... + (a_k - b_k)^2 + a_{k+1}^2 + a_{k+2}^2 + ... + a_n^2} \geq r \end{aligned}$$

That is, for every point $z \in F$, $d(y, z) \geq r$. Thus $B(y, r) \cap F = \emptyset \Rightarrow B(y, r) \subset F^c$. So F^c is open and F is closed.