2.37 (p. 25) (a) Let \( B = \{ x^2 + y^2 \leq 1 \} \) be a disk and \( p \) be a point on the circle \( \text{bd}(B) = \{ x^2 + y^2 = 1 \} \). Let \( S = B \setminus \{ p \} \). First, we show that \( S \) is convex. Let \( u, v \in S \) and \( w \in \text{relin}(xy) \). Since \( u, v \in B \) and \( B \) is convex, \( w \in B \). Obviously, \( w \not\in \text{bd}(B) \) and hence \( w \neq p \). Therefore, \( w \in S \) and \( \bar{uv} \subset S \). So \( S \) is convex. Next, we show that \( S \) is enclosed by \( B \). Let \( A \) be the set with \( S \subset A \subset B \). Then \( A \setminus S \subset B \setminus S = \{ p \} \). So \( A \setminus S = \emptyset \) or \( \{ p \} \). That is, \( A = S \) or \( B \). Therefore, there are no sets between \( S \) and \( B \) and \( S \) is enclosed by \( B \). Finally, we need to show that \( S \) is enclosed by exactly one convex set. Suppose that \( S \) is enclosed by a convex set \( A \). Let \( q \) be a point in \( A \) such that \( q \notin S \). If \( p = q \), then \( S \subset B \subset A \) and hence \( A = B \). Suppose that \( p \neq q \). Then \( q \notin B \). So \( d(o,q) > 1 \), where \( o \) is the origin. Since \( A \) is convex, \( \overline{pq} \subset A \), i.e., \( \lambda q \in A \) for every \( 1 \geq \lambda \geq 0 \). Let \( r = d(o,q) \). Then

\[
x = \frac{1 + r}{2r} q \in A
\]

but \( x \notin B \) because \( d(o,x) = ||x|| > 1 \). Then \( S \subset \text{conv}(S \cup \{ x \}) \subset A \) with \( S \neq \text{conv}(S \cup \{ x \}) \neq A \). Contradiction.

Similarly, we can construct convex sets enclosed by exactly \( n \) convex sets for all \( n \). Let \( G = \{ p_1, p_2, ..., p_n, ... \} \subset \text{bd}(B) \), \( A_n = B \setminus \{ p_1, p_2, ..., p_n \} \) and \( A = B \setminus G \).

Then \( A, A_1, A_2, ..., A_n, ... \) are convex; \( A_n \) is enclosed by exactly \( n \) convex sets, which are \( A_n \cup \{ p_1 \} \), \( A_n \cup \{ p_2 \} \), ..., and \( A_n \cup \{ p_n \} \); and \( A \) is enclosed by infinitely many convex sets, which are \( A \cup \{ p_1 \}, A \cup \{ p_2 \}, ..., A \cup \{ p_n \}, ... \).

(b) Let \( D = \{ x^2 + y^2 < 1 \} \) and \( G = \{ p_1, p_2, ..., p_n, ... \} \subset \text{bd}(D) \). Let \( A_i = D \cup \{ p_1, p_2, ..., p_i \} \). Then \( A_1 \subset A_2 \subset ... \subset A_i \subset A_{i+1} \subset ... \) with \( A_{i+1} \) enclosing \( A_i \).

(c) This is true. Suppose that there exist two distinct points \( p, q \notin A \) and \( p, q \notin B \). Then \( B \subset \text{conv}(B \cup \{ p \}) \subset A \). Since \( A \) encloses \( B \), there are no convex sets between \( B \) and \( A \) and hence \( \text{conv}(B \cup \{ p \}) = A \). Therefore, \( q \in \text{conv}(B \cup \{ p \}) \Rightarrow q \) is a convex combination of \( x_1, x_2, ..., x_{n-1}, p \) with \( x_1, x_2, ..., x_{n-1} \in B \). So there exist \( \lambda_1, \lambda_2, ..., \lambda_{n-1}, \lambda_n \geq 0 \) such that \( \sum_{i=1}^{n} \lambda_i = 1 \) and

\[
q = \lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_{n-1} x_{n-1} + \lambda_n p.
\]

We let

\[
y = \frac{1}{\lambda_1 + \lambda_2 + ... + \lambda_{n-1}} (\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_{n-1} x_{n-1})
\]

Then \( y \) is a convex combination of \( x_1, x_2, ..., x_{n-1} \) and hence \( y \in B \). And

\[
q = (\lambda_1 + \lambda_2 + ... + \lambda_{n-1}) y + \lambda_n p = (1 - \lambda_n) y + \lambda_n p
\]

and hence \( q \in \overline{pq} \) and \( q \neq p, y \).
By the same argument, we can show that \( p \in \overline{pq} \) for some \( x \in B \). Therefore,
\[
q = \alpha_1 p + \beta_1 y \\
p = \alpha_2 q + \beta_2 x
\]
where \( \alpha_i, \beta_i > 0 \) and \( \alpha_1 + \beta_1 = 1 \). We may eliminate \( q \) by multiplying the first equation by \( \alpha_2 \) and adding it to the second equation:
\[
p = \alpha_1 \alpha_2 p + \alpha_2 \beta_1 y + \beta_2 x
\]
So
\[
p = \frac{1}{1 - \alpha_1 \alpha_2} (\beta_2 x + \alpha_2 \beta_1 y)
\]
Since
\[
\beta_2 + \alpha_2 \beta_1 = 1 - \alpha_2 + \alpha_2 \beta_1 \\
= 1 - (1 - \beta_1) \alpha_2 = 1 - \alpha_1 \alpha_2
\]
p is a convex combination of \( x \) and \( y \), i.e., \( p \in \overline{xy} \). Then \( p \in B \), which is a contradiction.

(d) This is false. Let \( x_1, x_2, x_3 \) be three points on \( \mathbb{R}^2 \) not lying on a line and \( \Delta x_1 x_2 x_3 \) be the triangle with vertices at \( x_1, x_2, x_3 \) (including the boundary). Let \( B = (\Delta x_1 x_2 x_3 \setminus \overline{x_1 x_2}) \cup \{x_1\} \) (we remove the edge \( x_1 x_2 \) from the triangle but keeping one vertex \( x_1 \). We will show that \( B \) is convex but \( A = B \cup \{x_2\} \) is not.

To show that \( B \) is convex, let \( u, v \in B \) and we show that \( \overline{uv} \subset B \). Since \( u, v \in \Delta x_1 x_2 x_3 \),
\[
u = a_1 x_1 + a_2 x_2 + a_3 x_3 \\
v = b_1 x_1 + b_2 x_2 + b_3 x_3
\]
where \( a_i, b_i \geq 0 \) and \( a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = 1 \). Let \( w = \alpha u + \beta v \in \text{relint}(\overline{uv}) \), where \( \alpha, \beta > 0 \) and \( \alpha + \beta = 1 \). Then
\[
\alpha u + \beta v = (\alpha a_1 + \beta b_1) x_1 + (\alpha a_2 + \beta b_2) x_2 + (\alpha a_3 + \beta b_3) x_3
\]
If either \( a_3 > 0 \) or \( b_3 > 0 \), then \( \alpha a_3 + \beta b_3 > 0 \) and hence \( w \notin \overline{x_1 x_2} \); and since \( w \in \Delta x_1 x_2 x_3 \), \( w \in B \). Suppose that \( a_3 = b_3 = 0 \). Then \( u, v \in \overline{x_1 x_2} \).
And since \( u, v \in B \), \( u, v = x_1 \). Then \( w = x_1 \in B \). So \( B \) is convex.

Since \( x_1, x_2 \in A \) and \( \overline{x_1 x_2} \notin A \), \( A \) is not convex.

3.1 (p. 32) By the definition of normal vector, \( H = \{x : \langle u, x \rangle = \alpha\} \). Since \( x_1, x_2 \in H \), \( \langle u, x_1 \rangle = \langle u, x_2 \rangle = \alpha \). So \( \langle u, x_1 \rangle - \langle u, x_2 \rangle = 0 \) and \( \langle u, x_1 - x_2 \rangle = 0 \), i.e., \( u \) is orthogonal to \( x_1 - x_2 \).

3.2 (p. 32) Let \( e_1, e_2, ..., e_{n-1} \) be a basis for \( V \).

Since \( y \notin V \), \( \{e_1, e_2, ..., e_{n-1}, y\} \) is linearly independent and hence a basis for \( \mathbb{R}^n \). Therefore every point \( p \in \mathbb{R} \) is a linear combination of \( e_1, e_2, ..., e_{n-1}, y \), i.e.,
\[
p = a_1 e_1 + a_2 e_2 + ... + a_{n-1} e_{n-1} + ay
\]
Let \( x = a_1 e_1 + a_2 e_2 + ... + a_{n-1} e_{n-1} \). Then \( x \in V \) and \( p = x + ay \).
Next we show such representation is unique. Suppose that \( p = x + \alpha y = x' + \alpha' y \) with \( x, x' \in V \). Then \( x - x' = (\alpha' - \alpha)y \). If \( x = x' \), \( \alpha = \alpha' \). If \( x \neq x' \), \( \alpha \neq \alpha' \); then \( y = (\alpha' - \alpha)^{-1}(x - x') \in V \). \( \text{Contradiction.} \)

3.3 (p. 32) We want to show that \( x_0 + H_1 = [f : \beta] \).

Let \( x \in x_0 + H_1 \). \( x = x_0 + y \) with \( y \in H_1 \). Since \( y \in H_1 \), \( f(y) = \alpha \). So \( f(x) = f(x_0) + f(y) = f(x_0) + \alpha = \beta \). Therefore, \( x \in [f : \beta] \) and \( x_0 + H_1 \subseteq [f : \beta] \).

Let \( x \in [f : \beta] \). Then \( f(x) = \beta \). So \( f(x - x_0) = f(x) - f(x_0) = \beta - f(x_0) = \alpha \). Hence \( x - x_0 \in H_1 \) and \( x = x_0 + (x - x_0) \in x_0 + H_1 \). \( \text{So } [f : \beta] \subseteq x_0 + H_1 \).

In conclusion, \( x_0 + H_1 = [f : \beta] \).

3.4 (p. 32) (a) \( u = (2, -3) \)

(b) \((-1, 1) + H = \{2x - 3y = -3\} \).

3.5 (p. 32) We show that \( F_1 = F_2 \) if \( x_1 - x_2 \in V \) and \( F_1 \cap F_2 = \emptyset \).

Suppose that \( x_1 - x_2 \in V \). Then \( (x_1 - x_2) + V = V \Rightarrow x_1 + V = x_2 + V \), i.e., \( F_1 = F_2 \).

Suppose that \( x_1 - x_2 \notin V \). We want to show that \( F_1 \cap F_2 = \emptyset \). Otherwise, assume that \( F_1 \cap F_2 \neq \emptyset \). Let \( x \in F_1 \cap F_2 \). Since \( x \in F_1 = x_1 + V \), \( x - x_1 \in V \); since \( x \in F_2 = x_2 + V \), \( x - x_2 \in V \). And since \( V \) is a linear subspace, \( (x - x_2) - (x - x_1) \in V \), i.e., \( x_1 - x_2 \in V \). \( \text{Contradiction.} \) So \( F_1 \cap F_2 = \emptyset \).

3.6 (p. 33) \( L = \{x + 4y = 7\} \).

3.9 (p. 33) Let \( F = W + x \) where \( x \in \mathbb{R}^n \) and \( W \) is a linear subspace of \( \mathbb{R}^n \). There is nothing to prove if \( F = \mathbb{R}^n \). Let us assume that \( F \neq \mathbb{R}^n \). So \( W \) is a proper subspace of \( \mathbb{R}^n \).

Choose an orthonormal basis \( e_1, e_2, ..., e_n \) for \( \mathbb{R}^n \) with \( e_1, e_2, ..., e_k \) generating \( W \) and \( \|e_i\| = 1 \) for \( i = 1, 2, ..., n \).

Let \( y \notin F \) and \( y - x = a_1 e_1 + a_2 e_2 + ... + a_k e_k + a_{k+1} e_{k+1} + a_{k+2} e_{k+2} + ... + a_n e_n \). Since \( y - x \notin W \), \( a_{k+1}, a_{k+2}, ..., a_n \) are not all zero. Let

\[
r = \sqrt{a_{k+1}^2 + a_{k+2}^2 + ... + a_n^2}.
\]

Then \( r > 0 \).

Let \( z \in F \) and \( z = b_1 e_1 + b_2 e_2 + ... + b_k e_k \). Then

\[
||y - z|| = ||(a_1 - b_1)e_1 + (a_2 - b_2)e_2 + ... + (a_k - b_k)e_k + a_{k+1} e_{k+1} + a_{k+2} e_{k+2} + ... + a_n e_n||
= \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + ... + (a_k - b_k)^2 + a_{k+1}^2 + a_{k+2}^2 + ... + a_n^2} \geq r
\]

That is, for every point \( z \in F \), \( d(y, z) \geq r \). Thus \( B(y, r) \cap F = \emptyset \Rightarrow B(y, r) \subset F^c \). So \( F^c \) is open and \( F \) is closed.