Math 341 Homework 4 Solution

2.4 (p. 22) Note that \( \lambda B(x, \delta) = \{ \lambda y : \|y - x\| < \delta \} \) and \( B(\lambda x, \lambda \delta) = \{ z : \|z - \lambda x\| < \lambda \delta \} \).

Let \( \lambda y \in \lambda B(x, \delta) \). Then \( \|\lambda y - \lambda x\| = \lambda \|y - x\| < \lambda \delta \). So \( \lambda y \in B(\lambda x, \lambda \delta) \) and \( \lambda B(x, \delta) \subset B(\lambda x, \lambda \delta) \).

On the other hand, let \( z \in B(\lambda x, \lambda \delta) \). Then
\[
\|\lambda^{-1} z - x\| = \frac{1}{\lambda} \|z - \lambda x\| < \delta
\]
So \( \lambda^{-1} z \in B(x, \delta) \) and \( z \in \lambda B(x, \delta) \). Therefore, \( \lambda B(x, \delta) \supset B(\lambda x, \lambda \delta) \).

In conclusion, \( \lambda B(x, \delta) = B(\lambda x, \lambda \delta) \).

2.17 (p. 23) Pick three points \( x_1, x_2, x_3 \in \mathbb{R}^2 \) not lying on a line. Let \( A = \overline{x_1x_2} \) and \( B = \overline{x_1x_2x_3} \) (including the boundary), i.e.,
\[
B = \{ \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 : \lambda_1 + \lambda_2 + \lambda_3 = 1, \lambda_1, \lambda_2, \lambda_3 \geq 0 \}
\]
Obvious, \( A \subset B \) is one side of the triangle. But
\[
\text{relint}(A) = \overline{x_1x_2} \setminus \{x_1, x_2\}
\]
and
\[
\text{relint}(B) = B \setminus (\overline{x_1x_2} \cup \overline{x_2x_3} \cup \overline{x_3x_1})
\]
That is, \( \text{relint}(A) \) is the line segment \( \overline{x_1x_2} \) with the two ending points removed and \( \text{relint}(B) \) is the triangle with three sides removed. Of course, \( \text{relint}(A) \not\subset \text{relint}(B) \).

2.18 (p. 23) If \( \overline{xy} \subset \text{bd}(S) \), there is nothing to prove. Otherwise, suppose that \( \overline{xy} \not\subset \text{bd}(S) \), there exists a point \( z \in \text{relint} \overline{xy} \) such that \( z \not\in \text{bd}(S) \).

And since \( z \in S \), \( z \in \text{int}(S) \). By 2.12, \( x \in \text{bd}(S) \) and \( z \in \text{int}(S) \) \( \Rightarrow \) \( \text{relint} \overline{xy} \subset \text{int}(S) \) and \( y \in \text{bd}(S) \) and \( z \in \text{int}(S) \) \( \Rightarrow \) \( \text{relint} \overline{yz} \subset \text{int}(S) \).

Therefore, \( \text{relint} \overline{xy} \subset \text{int}(S) \).

2.20 (p. 23) First, I want state and prove a simple fact: If \( x, y, z \) are three distinct points on a line, then \( x \in \text{aff} \{y, z\} \), \( y \in \text{aff} \{x, z\} \) and \( z \in \text{aff} \{x, y\} \).

Without the loss of generality, let us assume that \( y \in \overline{xy} \). So \( y = (1 - \lambda)x + \lambda z \) for some \( 0 < \lambda < 1 \). Obviously, \( y \) is an affine combination of \( x \) and \( z \) so \( y \in \text{aff} \{z, x\} \). Since
\[
z = \frac{1}{\lambda} y - \frac{1 - \lambda}{\lambda} x \text{ and } \frac{1}{\lambda} - \frac{1 - \lambda}{\lambda} = 1
\]
\( z \in \text{aff} \{x, y\} \). Similarly,
\[
x = \frac{1}{1 - \lambda} y - \frac{\lambda}{1 - \lambda} z \Rightarrow x \in \text{aff} \{y, z\}
\]
Now let us go back to the proof of the problem. Suppose that \( y \in \text{aff}(S) = W \). Since \( x \in \text{relint}(S) \), there exists an open ball \( B(x, r) \) such that \( B(x, r) \cap W \subset \text{relint}(S) \). Obviously, \( \overline{xy} \subset W \) and \( \text{relint}(\overline{xy}) \cap B(x, r) \neq \emptyset \). Therefore, \( \text{relint}(\overline{xy}) \cap B(x, r) \cap W \neq \emptyset \) and hence \( \text{relint}(\overline{xy}) \cap \text{relint}(S) \neq \emptyset \).

On the other hand, suppose that \( \text{relint}(\overline{xy}) \cap \text{relint}(S) \neq \emptyset \). Let \( z \in \text{relint}(\overline{xy}) \cap \text{relint}(S) \). Since \( x, y, z \) lie on the same line, \( y \in \text{aff} \{x, z\} \subset \text{aff}(S) \).
2.25 (p. 23) (a) \( \text{conv}\{x_1, x_2, x_3\} \) is the triangle \( \Delta x_1 x_2 x_3 \) and \( \text{pos}\{x_1, x_2, x_3\} \) is the cone over \( \Delta x_1 x_2 x_3 \) with vertex at the origin. So \( \text{pos}\{x_1, x_2, x_3\} = \{(x, y) : x \geq 0, y \geq 0\} \).

Since \( \{x_1, x_2, x_3\} \) is affinely independent, \( \text{aff}\{x_1, x_2, x_3\} = \mathbb{R}^2 \). So \( x \in \text{pos}\{x_1, x_2, x_3\} \cap \text{aff}\{x_1, x_2, x_3\} \) while \( x \notin \text{conv}\{x_1, x_2, x_3\} \).

(b) \( S \) is linearly independent.

2.26 (p. 24) (a) Since \( S \subset \text{conv}(S) \), \( \text{pos}(S) \subset \text{pos}(\text{conv}(S)) \). It suffices to prove that \( \text{pos}(\text{conv}(S)) \subset \text{pos}(S) \).

Let \( x \in \text{pos}(\text{conv}(S)) \). Then by (b), \( x = \lambda s \) for some \( \lambda \geq 0 \) and \( s \in \text{conv}(S) \). Since \( s \in \text{conv}(S) \), there exist \( x_1, x_2, ..., x_n \in S \) and \( a_1, a_2, ..., a_n \geq 0 \) such that \( s = a_1 x_1 + a_2 x_2 + ... + a_n x_n \) and \( \sum_{i=1}^n a_i = 1 \). So \( x = \sum_{i=1}^n a_i x_i \).

And since \( \lambda a_i \geq 0, x \in \text{pos}(S) \). Therefore, \( \text{pos}(\text{conv}(S)) \subset \text{pos}(S) \) and hence \( \text{pos}(\text{conv}(S)) = \text{pos}(S) \).

(b) Suppose that \( x \in \text{pos}(S) \). Then \( x = \sum_{i=1}^n a_i x_i \) for some \( a_1, a_2, ..., a_n \geq 0 \) and \( x_1, x_2, ..., x_n \in S \).

If \( a_1 = a_2 = ... = a_n = 0 \), then \( x = 0 = 0s \) for every \( s \in S \). Assume that \( a_1, a_2, ..., a_n \) are not all zero. Let \( \lambda = \sum_{i=1}^n a_i \). Then \( \lambda^{-1} x = \sum_{i=1}^n \lambda a_i^{-1} x_i \) is a convex combination of \( x_1, x_2, ..., x_n \) since \( \sum_{i=1}^n a_i = 1 \).

And since \( S \) is convex, \( \lambda^{-1} x \in S \). Let \( s = \lambda^{-1} x \). Then \( x = \lambda s \) with \( \lambda \geq 0 \) and \( s \in S \).

On the other hand, it is obvious that \( x = \lambda s \) with \( \lambda \geq 0 \) and \( s \in S \Rightarrow x \in \text{pos}(S) \).

2.30 (p. 24) (a) Check that the following matrix has rank 4 (use Gaussian reduction)

\[
\begin{bmatrix}
1 & -1 & 2 & -1 & 1 \\
2 & -1 & 2 & 0 & 1 \\
1 & 0 & 2 & 0 & 1 \\
1 & 0 & 3 & 1 & 1 \\
\end{bmatrix}
\]

(b) It is easy to check that \( x_i \in B \) for \( i = 1, 2, 3, 4 \). So \( A \subset B \). By (a), \( \dim A = 3 \). And since \( \dim B = 3 = \dim A \), \( A = B \) by 2.29.

2.31 (p. 24) (a) Suppose that \( F \notin G \) and \( G \notin F \). Then there exist \( x \in F \) and \( y \in G \) such that \( x \notin G \) and \( y \notin F \). Choose a point \( z \in \overline{xy} \) and \( z \neq x, y \).

Since \( F \cup G \) is convex, \( z \in F \cup G \). Since \( x, y, z \) lies on a line, \( x \in \text{aff}\{y, z\} \) and \( y \in \text{aff}\{x, z\} \). If \( z \in F \), then \( y \in \text{aff}\{x, z\} \subset F \); otherwise, if \( z \in G \), then \( x \in \text{aff}\{y, z\} \subset G \). Either way we have a contradiction.

(b) Take \( A = (0, 2) \) and \( B = (1, 3) \subset \mathbb{R} \). Then \( A, B, A \cup B \) are all convex but \( A \not\subset B \) and \( B \not\subset A \).

2.32 (p. 24) Suppose that \( x_1, x_2, ..., x_k \) are linearly dependent. Then there exist \( a_1, a_2, ..., a_k \) not all zero such that \( a_1 x_1 + a_2 x_2 + ... + a_k x_k = 0 \). Without the loss of generality, assume that \( a_1 \neq 0 \). Then

\[
a_1 \langle x_1, x_1 \rangle + a_2 \langle x_2, x_1 \rangle + ... + a_k \langle x_k, x_1 \rangle = 0
\]

Since \( x_1, x_2, ..., x_k \) are orthogonal to each other, \( \langle x_2, x_1 \rangle = ... = \langle x_k, x_1 \rangle = 0 \).

So \( a_1 \langle x_1, x_1 \rangle = 0 \Rightarrow \langle x_1, x_1 \rangle = 0 \Rightarrow x_1 = 0 \). Contradiction.
2.34 (p. 25) By Caratheodory’s Theorem, \( x \in \{x_1, x_2, \ldots, x_{n+1}\} \) for some \( x_1, x_2, \ldots, x_{n+1} \in S \). Let
\[
x = a_1x_1 + a_2x_2 + \ldots + a_{n+1}x_{n+1}
\]
with \( a_i \geq 0 \) and \( \sum_{i=1}^{n+1} a_i = 1 \).

Since \( \{v, x_1, x_2, \ldots, x_{n+1}\} \) is affinely dependent, there exist \( b, b_1, b_2, \ldots, b_{n+1}, \) not all zero, such that
\[

bv + b_1x_1 + b_2x_2 + \ldots + b_{n+1}x_{n+1} = 0
\]
and \( b + \sum_{i=1}^{n+1} b_i = 0 \). We may choose \( b \) such that \( b \leq 0 \) (otherwise, we replace \( b, b_i \) by \( -b, -b_i \)).

Let
\[

\mu = \min \{ \frac{a_i}{b_i} : b_i > 0 \}
\]
Then
\[
x = a_1x_1 + a_2x_2 + \ldots + a_{n+1}x_{n+1} - \mu(bv + b_1x_1 + b_2x_2 + \ldots + b_{n+1}x_{n+1})
\]
\[
\quad = -\mu bv + \sum_{i=1}^{n+1} (a_i - \mu b_i)x_i
\]

Obviously, \( -\mu b + \sum_{i=1}^{n+1} (a_i - \mu b_i) = \sum_{i=1}^{n+1} a_i - \mu(b + \sum_{i=1}^{n+1} b_i) = 1 \). And
\(-\mu b \geq 0, a_i - \mu b_i \geq 0 \) due to our choice of \( \mu \). At least one of \( a_i - \mu b_i \) is zero.

Let \( \mu = a_i/b_i \) for some \( 1 \leq l \leq n+1 \). Then \( x \in \text{conv}\{v, x_1, x_2, \ldots, x_{n+1}\} \).

**A1.** Proof by induction. It holds when \( n = 2 \) (Theorem 2.9). Suppose that it holds for \( n < k \). We want to prove it for \( n = k \).

Let \( x \in \text{conv}\{x_1, x_2, \ldots, x_k\} \) with \( x_1, x_2, \ldots, x_k \in \text{int}(S) \). Let \( x = a_1x_1 + a_2x_2 + \ldots + a_kx_k \) with \( a_i \geq 0 \) and \( \sum_{i=1}^{k} a_i = 1 \). Without the loss of generality, assume that \( a_1 > 0 \). Let \( a = \sum_{i=1}^{k-1} a_i = 1 - a_k \). We write
\[
x = (a_1x_1 + a_2x_2 + \ldots + a_{k-1}x_{k-1}) + a_kx_k
\]
\[
= a\left(\frac{a_1}{a}x_1 + \frac{a_2}{a}x_2 + \ldots + \frac{a_{k-1}}{a}x_{k-1}\right) + (1 - a)x_k
\]
\[
= ay + (1-a)x_k
\]
Let
\[
y = \frac{a_1}{a}x_1 + \frac{a_2}{a}x_2 + \ldots + \frac{a_{k-1}}{a}x_{k-1}.
\]
Since
\[
\frac{a_1}{a} + \frac{a_2}{a} + \ldots + \frac{a_{k-1}}{a} = \frac{a}{a} = 1
\]
y \( \in \text{conv}\{x_1, x_2, \ldots, x_{k-1}\} \). By induction hypothesis, \( y \in \text{int}(S) \). Since
\( x = ay + (1-a)x_k \), \( x \in \text{conv}\{y, x_k\} \). Therefore, \( x \in \text{int}(S) \).

**A2.** Since \( S \) is closed, \( S \) has a minimum \( m \) if \( S \) is bounded from below and \( S \) has a maximum \( M \) if \( S \) is bounded from above. Then it is obvious that \( \text{conv}(S) \) can only be one of the following: \([m, M]\), \([m, \infty)\), \((\infty, M]\) or \((-\infty, \infty)\). All these are closed sets.