

Math 341 Homework 4 Solution

2.4 (p. 22) Note that $\lambda B(x, \delta) = \{\lambda y : \|y - x\| < \delta\}$ and $B(\lambda x, \lambda\delta) = \{z : \|z - \lambda x\| < \lambda\delta\}$.

Let $\lambda y \in \lambda B(x, \delta)$. Then $\|\lambda y - \lambda x\| = \lambda\|y - x\| < \lambda\delta$. So $\lambda y \in B(\lambda x, \lambda\delta)$ and $\lambda B(x, \delta) \subset B(\lambda x, \lambda\delta)$.

On the other hand, let $z \in B(\lambda x, \lambda\delta)$. Then

$$\|\lambda^{-1}z - x\| = \frac{1}{\lambda}\|z - \lambda x\| < \delta$$

So $\lambda^{-1}z \in B(x, \delta)$ and $z \in \lambda B(x, \delta)$. Therefore, $\lambda B(x, \delta) \supset B(\lambda x, \lambda\delta)$.

In conclusion, $\lambda B(x, \delta) = B(\lambda x, \lambda\delta)$.

2.17 (p. 23) Pick three points $x_1, x_2, x_3 \in \mathbb{R}^2$ not lying on a line. Let $A = \overline{x_1x_2}$ and B be the triangle $\Delta x_1x_2x_3$ (including the boundary), i.e.,

$$B = \{\lambda_1x_1 + \lambda_2x_2 + \lambda_3x_3 : \lambda_1 + \lambda_2 + \lambda_3 = 1, \lambda_1, \lambda_2, \lambda_3 \geq 0\}$$

Obviously, $A \subset B$ is one side of the triangle. But

$$\text{relint}(A) = \overline{x_1x_2} \setminus \{x_1, x_2\}$$

and

$$\text{relint}(B) = B \setminus (\overline{x_1x_2} \cup \overline{x_2x_3} \cup \overline{x_3x_1})$$

That is, $\text{relint}(A)$ is the line segment $\overline{x_1x_2}$ with the two ending points removed and $\text{relint}(B)$ is the triangle with three sides removed. Of course, $\text{relint}(A) \not\subset \text{relint}(B)$.

2.18 (p. 23) If $\overline{xy} \subset \text{bd}(S)$, there is nothing to prove. Otherwise, suppose that $\overline{xy} \not\subset \text{bd}(S)$, there exists a point $z \in \text{relint } \overline{xy}$ such that $z \notin \text{bd}(S)$. And since $z \in S$, $z \in \text{int}(S)$. By 2.12, $x \in \text{bd}(S)$ and $z \in \text{int}(S) \Rightarrow \text{relint } \overline{xz} \subset \text{int}(S)$ and $y \in \text{bd}(S)$ and $z \in \text{int}(S) \Rightarrow \text{relint } \overline{yz} \subset \text{int}(S)$. Therefore, $\text{relint } \overline{xy} \subset \text{int}(S)$.

2.20 (p. 23) First, I want state and prove a simple fact: If x, y, z are three distinct points on a line, then $x \in \text{aff}\{y, z\}$, $y \in \text{aff}\{z, x\}$ and $z \in \text{aff}\{x, y\}$.

Without the loss of generality, let us assume that $y \in \overline{xz}$. So $y = (1 - \lambda)x + \lambda z$ for some $0 < \lambda < 1$. Obviously, y is an affine combination of x and z so $y \in \text{aff}\{z, x\}$. Since

$$z = \frac{1}{\lambda}y - \frac{1 - \lambda}{\lambda}x \text{ and } \frac{1}{\lambda} - \frac{1 - \lambda}{\lambda} = 1$$

$z \in \text{aff}\{x, y\}$. Similarly,

$$x = \frac{1}{1 - \lambda}y - \frac{\lambda}{1 - \lambda}z \Rightarrow x \in \text{aff}\{y, z\}$$

Now let us go back to the proof of the problem. Suppose that $y \in \text{aff}(S) = W$. Since $x \in \text{relint}(S)$, there exists an open ball $B(x, r)$ such that $B(x, r) \cap W \subset \text{relint}(S)$. Obviously, $\overline{xy} \subset W$ and $\text{relint}(\overline{xy}) \cap B(x, r) \neq \emptyset$. Therefore, $\text{relint}(\overline{xy}) \cap B(x, r) \cap W \neq \emptyset$ and hence $\text{relint}(\overline{xy}) \cap \text{relint}(S) \neq \emptyset$.

On the otherhand, suppose that $\text{relint}(\overline{xy}) \cap \text{relint}(S) \neq \emptyset$. Let $z \in \text{relint}(\overline{xy}) \cap \text{relint}(S)$. Since x, y, z lie on the same line, $y \in \text{aff}\{x, z\} \subset \text{aff}(S)$.

2.25 (p. 23) (a) $\text{conv}\{x_1, x_2, x_3\}$ is the triangle $\Delta x_1 x_2 x_3$ and $\text{pos}\{x_1, x_2, x_3\}$ is the cone over $\Delta x_1 x_2 x_3$ with vertex at the origin. So $\text{pos}\{x_1, x_2, x_3\} = \{(x, y) : x \geq 0, y \geq 0\}$.

Since $\{x_1, x_2, x_3\}$ is affinely independent, $\text{aff}\{x_1, x_2, x_3\} = \mathbb{R}^2$. So $x \in \text{pos}\{x_1, x_2, x_3\} \cap \text{aff}\{x_1, x_2, x_3\}$ while $x \notin \text{conv}\{x_1, x_2, x_3\}$.

(b) S is linearly independent.

2.26 (p. 24) (a) Since $S \subset \text{conv}(S)$, $\text{pos}(S) \subset \text{pos}(\text{conv}(S))$. It suffices to prove that $\text{pos}(\text{conv}(S)) \subset \text{pos}(S)$.

Let $x \in \text{pos}(\text{conv}(S))$. Then by (b), $x = \lambda s$ for some $\lambda \geq 0$ and $s \in \text{conv}(S)$. Since $s \in \text{conv}(S)$, there exist $x_1, x_2, \dots, x_n \in S$ and $a_1, a_2, \dots, a_n \geq 0$ such that $s = a_1 x_1 + a_2 x_2 + \dots + a_n x_n$ and $\sum_{i=1}^n a_i = 1$. So $x = \sum_{i=1}^n \lambda a_i x_i$. And since $\lambda a_i \geq 0$, $x \in \text{pos}(S)$. Therefore, $\text{pos}(\text{conv}(S)) \subset \text{pos}(S)$ and hence $\text{pos}(\text{conv}(S)) = \text{pos}(S)$.

(b) Suppose that $x \in \text{pos}(S)$. Then $x = \sum_{i=1}^n a_i x_i$ for some $a_1, a_2, \dots, a_n \geq 0$ and $x_1, x_2, \dots, x_n \in S$.

If $a_1 = a_2 = \dots = a_n = 0$, then $x = 0 = 0s$ for every $s \in S$. Assume that a_1, a_2, \dots, a_n are not all zero. Let $\lambda = \sum_{i=1}^n a_i$. Then $\lambda^{-1}x = \sum_{i=1}^n a_i \lambda^{-1} x_i$ is a convex combination of x_1, x_2, \dots, x_n since $\sum_{i=1}^n a_i \lambda^{-1} = \lambda^{-1} \sum_{i=1}^n a_i = 1$. And since S is convex, $\lambda^{-1}x \in S$. Let $s = \lambda^{-1}x$. Then $x = \lambda s$ with $\lambda \geq 0$ and $s \in S$.

On the otherhand, it is obvious that $x = \lambda s$ with $\lambda \geq 0$ and $s \in S \Rightarrow x \in \text{pos}(S)$.

2.30 (p. 24) (a) Check that the following matrix has rank 4 (use Gaussian reduction)

$$\begin{bmatrix} 1 & -1 & 2 & -1 & 1 \\ 2 & -1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 3 & 1 & 1 \end{bmatrix}$$

(b) It is easy to check that $x_i \in B$ for $i = 1, 2, 3, 4$. So $A \subset B$. By (a), $\dim A = 3$. And since $\dim B = 3 = \dim A$, $A = B$ by 2.29.

2.31 (p. 24) (a) Suppose that $F \not\subset G$ and $G \not\subset F$. Then there exist $x \in F$ and $y \in G$ such that $x \notin G$ and $y \notin F$. Choose a point $z \in \overline{xy}$ and $z \neq x, y$. Since $F \cup G$ is convex, $z \in F \cup G$. Since x, y, z lies on a line, $x \in \text{aff}\{y, z\}$ and $y \in \text{aff}\{x, z\}$. If $z \in F$, then $y \in \text{aff}\{x, z\} \subset F$; otherwise, if $z \in G$, then $x \in \text{aff}\{y, z\} \subset G$. Either way we have a contradiction.

(b) Take $A = (0, 2)$ and $B = (1, 3) \subset \mathbb{R}$. Then $A, B, A \cup B$ are all convex but $A \not\subset B$ and $B \not\subset A$.

2.32 (p. 24) Suppose that x_1, x_2, \dots, x_k are linearly dependent. Then there exist a_1, a_2, \dots, a_k not all zero such that $a_1 x_1 + a_2 x_2 + \dots + a_k x_k = 0$. Without the loss of generality, assume that $a_1 \neq 0$. Then

$$a_1 \langle x_1, x_1 \rangle + a_2 \langle x_2, x_1 \rangle + \dots + a_k \langle x_k, x_1 \rangle = 0$$

Since x_1, x_2, \dots, x_k are orthogonal to each other, $\langle x_2, x_1 \rangle = \dots = \langle x_k, x_1 \rangle = 0$. So $a_1 \langle x_1, x_1 \rangle = 0 \Rightarrow \langle x_1, x_1 \rangle = 0 \Rightarrow x_1 = 0$. Contradiction.

2.34 (p. 25) By Caratheodory's Theorem, $x \in \{x_1, x_2, \dots, x_{n+1}\}$ for some $x_1, x_2, \dots, x_{n+1} \in S$. Let

$$x = a_1x_1 + a_2x_2 + \dots + a_{n+1}x_{n+1}$$

with $a_i \geq 0$ and $\sum_{i=1}^{n+1} a_i = 1$.

Since $\{v, x_1, x_2, \dots, x_{n+1}\}$ is affinely dependent, there exist $b, b_1, b_2, \dots, b_{n+1}$, not all zero, such that

$$bv + b_1x_1 + b_2x_2 + \dots + b_{n+1}x_{n+1} = 0$$

and $b + \sum_{i=1}^{n+1} b_i = 0$. We may choose b such that $b \leq 0$ (otherwise, we replace b, b_i by $-b, -b_i$).

Let

$$\mu = \min\left\{\frac{a_i}{b_i} : b_i > 0\right\}$$

Then

$$\begin{aligned} x &= a_1x_1 + a_2x_2 + \dots + a_{n+1}x_{n+1} - \mu(bv + b_1x_1 + b_2x_2 + \dots + b_{n+1}x_{n+1}) \\ &= -\mu bv + \sum_{i=1}^{n+1} (a_i - \mu b_i)x_i \end{aligned}$$

Obviously, $-\mu b + \sum_{i=1}^{n+1} (a_i - \mu b_i) = \sum_{i=1}^{n+1} a_i - \mu(b + \sum_{i=1}^{n+1} b_i) = 1$. And $-\mu b \geq 0$, $a_i - \mu b_i \geq 0$ due to our choice of μ . At least one of $a_i - \mu b_i$ is zero.

Let $\mu = a_l/b_l$ for some $1 \leq l \leq n+1$. Then $x \in \text{conv}\{v, x_1, x_2, \dots, \hat{x}_l, \dots, x_{n+1}\}$.

A1. Proof by induction. It holds when $n = 2$ (Theorem 2.9). Suppose that it holds for $n < k$. We want to prove it for $n = k$.

Let $x \in \text{conv}\{x_1, x_2, \dots, x_k\}$ with $x_1, x_2, \dots, x_k \in \text{int}(S)$. Let $x = a_1x_1 + a_2x_2 + \dots + a_kx_k$ with $a_i \geq 0$ and $\sum_{i=1}^k a_i = 1$. Without the loss of generality, assume that $a_1 > 0$. Let $a = \sum_{i=1}^{k-1} a_i = 1 - a_k$. We write

$$\begin{aligned} x &= (a_1x_1 + a_2x_2 + \dots + a_{k-1}x_{k-1}) + a_kx_k \\ &= a\left(\frac{a_1}{a}x_1 + \frac{a_2}{a}x_2 + \dots + \frac{a_{k-1}}{a}x_{k-1}\right) + (1-a)x_k \\ &= ay + (1-a)x_k \end{aligned}$$

Let

$$y = \frac{a_1}{a}x_1 + \frac{a_2}{a}x_2 + \dots + \frac{a_{k-1}}{a}x_{k-1}.$$

Since

$$\frac{a_1}{a} + \frac{a_2}{a} + \dots + \frac{a_{k-1}}{a} = \frac{a}{a} = 1$$

$y \in \text{conv}\{x_1, x_2, \dots, x_{k-1}\}$. By induction hypothesis, $y \in \text{int}(S)$. Since $x = ay + (1-a)x_k$, $x \in \text{conv}\{y, x_k\}$. Therefore, $x \in \text{int}(S)$.

A2. Since S is closed, S has a minimum m if S is bounded from below and S has a maximum M if S is bounded from above. Then it is obvious that $\text{conv}(S)$ can only be one of the following: $[m, M]$, $[m, \infty)$, $(-\infty, M]$ or $(-\infty, \infty)$. All these are closed sets.