

Math 341 Homework 4 (due Feb. 13)

Problems from the book: (pp. 23-25) 2.4, 2.17, 2.18, 2.20, 2.25, 2.26, 2.30, 2.31, 2.32, 2.34

Additional problems:

A1. Let  $S$  be a convex set and  $x_1, x_2, \dots, x_n \in \text{int}(S)$ . Show that

$$\text{conv}\{x_1, x_2, \dots, x_n\} \subset \text{int}(S).$$

A2. We have seen that it is not true that  $S$  is closed  $\Rightarrow$   $\text{conv}(S)$  is closed.

However, this is true in dimension one, i.e., prove the following: Let  $S$  be a closed subset of  $\mathbb{R}$ . Then  $\text{conv}(S)$  is also closed.

**Hints**

2.17 The statement is false. Pick three points  $x_1, x_2, x_3 \in \mathbb{R}^2$  not lying on a line. Let  $A = \overline{x_1x_2}$  and  $B$  be the triangle  $\Delta x_1x_2x_3$  (including the boundary), i.e.,

$$B = \{\lambda_1x_1 + \lambda_2x_2 + \lambda_3x_3 : \lambda_1 + \lambda_2 + \lambda_3 = 1, \lambda_1, \lambda_2, \lambda_3 \geq 0\}$$

Show that  $A \subset B$  but  $\text{relint}(A) \not\subset \text{relint}(B)$ .

2.18 If  $\overline{xy} \not\subset \text{bd}(S)$ , then there exists a point  $z \in \overline{xy}$  such that  $z \in \text{int}(S)$ . Use 2.12 to finish the argument.

2.20 You need to prove it in two directions “ $\Rightarrow$ ” and “ $\Leftarrow$ ”. I will only give hints for “ $\Leftarrow$ ” as “ $\Rightarrow$ ” is trivial. Let  $z \in \text{relint}(\overline{xy}) \cap \text{relint}(S)$ . Show that  $y$  is an affine combination of  $x$  and  $z$ .

2.25 For (b), the special property that  $S$  has is that  $S$  is linearly independent. Show that if  $S$  is a set of vectors which are linearly independent, then  $\text{pos}(S) \cap \text{aff}(S) = \text{conv}(S)$ .

2.31 Again you have to prove it in two directions “ $\Rightarrow$ ” and “ $\Leftarrow$ ”. I will give hints for “ $\Rightarrow$ ” as “ $\Leftarrow$ ” is trivial. Proof by contradiction: Assume the opposite, i.e., there exists  $x \in F$  and  $y \in G$  such that  $x \notin G$  and  $y \notin F$ . Since  $F \cup G$  is convex,  $\overline{xy} \subset F \cup G$ . Let  $z$  be a point on  $\overline{xy}$  with  $z \neq x, y$ . Show that  $x$  is an affine combination of  $y, z$  and  $y$  is an affine combination of  $x, z$ . Then use the fact  $z \in F \cup G$  to finish the proof. For (b), you can find an example in dimension one, i.e.,  $F, G \subset \mathbb{R}$  convex,  $F \not\subset G$ ,  $G \not\subset F$  and  $F \cup G$  convex.

2.32 Two vectors are orthogonal if their inner product is zero. Check any linear algebra textbook if you do not know how to prove it.

2.34 Assume that  $v \in \text{conv}\{x_1, x_2, \dots, x_{n+1}\}$ . Then  $x_1, x_2, \dots, x_{n+1}, v$  are affinely dependent. Divide it into two cases: (1)  $x_1, x_2, \dots, x_{n+1}$  are affinely dependent; (2)  $x_1, x_2, \dots, x_{n+1}$  are affinely independent. In each case, use the same argument as in the proof of the original Caratheodory theorem. This is the hardest problem in this assignment. Good luck.