Math 341 Homework 4 (due Feb. 13)

Problems from the book: (pp. 23-25) 2.4, 2.17, 2.18, 2.20, 2.25, 2.26, 2.30, 2.31, 2.32, 2.34

Additional problems:
A1. Let $S$ be a convex set and $x_1, x_2, ..., x_n \in \text{int}(S)$. Show that
$$\text{conv}\{x_1, x_2, ..., x_n\} \subset \text{int}(S).$$

A2. We have seen that it is not true that $S$ is closed $\Rightarrow \text{conv}(S)$ is closed.
However, this is true in dimension one, i.e., prove the following: Let $S$ be a closed subset of $\mathbb{R}$. Then $\text{conv}(S)$ is also closed.

Hints

2.17 The statement is false. Pick three points $x_1, x_2, x_3 \in \mathbb{R}^2$ not lying on a line. Let $A = \overrightarrow{x_1x_2}$ and $B$ be the triangle $\Delta x_1x_2x_3$ (including the boundary), i.e.,
$$B = \{\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 : \lambda_1 + \lambda_2 + \lambda_3 = 1, \lambda_1, \lambda_2, \lambda_3 \geq 0\}$$

Show that $A \subset B$ but $\text{relint}(A) \not\subset \text{relint}(B)$.

2.18 If $x\overline{y} \not\subset \text{bd}(S)$, then there exists a point $z \in x\overline{y}$ such that $z \in \text{int}(S)$. Use 2.12 to finish the argument.

2.20 You need to prove it in two directions “$\Rightarrow$” and “$\Leftarrow$”. I will only give hints for “$\Leftarrow$” as “$\Rightarrow$” is trivial. Let $z \in \text{relint}(x\overline{y}) \cap \text{relint}(S)$. Show that $y$ is an affine combination of $x$ and $z$.

2.25 For (b), the special property that $S$ has is that $S$ is linearly independent. Show that if $S$ is a set of vectors which are linearly independent, then $\text{pos}(S) \cap \text{aff}(S) = \text{conv}(S)$.

2.31 Again you have to prove it in two directions “$\Rightarrow$” and “$\Leftarrow$”. I will give hints for “$\Rightarrow$” as “$\Leftarrow$” is trivial. Proof by contradiction: Assume the opposite, i.e., there exists $x \in F$ and $y \in G$ such that $x \not\in G$ and $y \not\in F$. Since $F \cup G$ is convex, $x\overline{y} \subset F \cup G$. Let $z$ be a point on $x\overline{y}$ with $z \neq x, y$. Show that $x$ is an affine combination of $y, z$ and $y$ is an affine combination of $x, z$. Then use the fact $z \in F \cup G$ to finish the proof. For (b), you can find an example in dimension one, i.e., $F, G \subset \mathbb{R}$ convex, $F \not\subset G$, $G \not\subset F$ and $F \cup G$ convex.

2.32 Two vectors are orthogonal if their inner product is zero. Check any linear algebra textbook if you do not know how to prove it.

2.34 Assume that $v \in \text{conv}\{x_1, x_2, ..., x_{n+1}\}$. Then $x_1, x_2, ..., x_{n+1}, v$ are affinely dependent. Divide it into two cases: (1) $x_1, x_2, ..., x_{n+1}$ are affinely dependent; (2) $x_1, x_2, ..., x_{n+1}$ are affinely independent. In each case, use the same argument as in the proof of the original Caratheodory theorem. This is the hardest problem in this assignment. Good luck.