Math 341 Homework 3 Solution

2.13 (p. 23) Fix a point $y \in \operatorname{int}(S)$. For a point $x \in \operatorname{cl}(S)$, $\operatorname{relint}(\overline{xy}) \subset \operatorname{int}(S)$ by 2.12. Therefore,

$$\overline{xy} = \operatorname{cl}(\operatorname{relint}(\overline{xy})) \subset \operatorname{int}(S)$$

So $x \in \operatorname{cl}(\operatorname{int}(S))$ and $\operatorname{cl}(S) \subset \operatorname{cl}(\operatorname{int}(S))$. And since $\operatorname{cl}(S) \supset \operatorname{cl}(\operatorname{int}(S))$, $\operatorname{cl}(S) = \operatorname{cl}(\operatorname{int}(S))$.

Consider $S = \{2\} \cup (0,1) \subset \mathbb{R}$. We have $\operatorname{int}(S) = (0,1)$ and $\operatorname{cl}(\operatorname{int}(S)) = [0,1]$ while $\operatorname{cl}(S) = \{2\} \cup [0,1]$.

2.14 (p. 23) Let $A = \operatorname{int}(\operatorname{cl}(S))$ and $B = \operatorname{int}(S)$. By 2.13, $\operatorname{cl}(A) = \operatorname{cl}(\operatorname{int}(\operatorname{cl}(S))) = \operatorname{cl}(\operatorname{cl}(S)) = \operatorname{cl}(S)$ and $\operatorname{cl}(B) = \operatorname{cl}(\operatorname{int}(S)) = \operatorname{cl}(S)$. So $\operatorname{cl}(A) = \operatorname{cl}(B)$. To prove $A = B$, it suffices to prove the following: if A and B are two open convex sets satisfying $\operatorname{cl}(A) = \operatorname{cl}(B)$, then $A = B$.

Let $x \in A$ and $y \in B$. Since A is open, x is an interior point of A. Hence there exists an open ball $B(x,\delta) \subset A$ with $\delta > 0$. Let $r = d(x,y)$,

$$\lambda = \frac{2r}{2r+\delta} \text{ and } z = \frac{1}{\lambda}(x - (1-\lambda)y)$$

One can check that $x = (1-\lambda)y + \lambda z$. So $x \in \operatorname{relint}(\overline{yz})$. Since

$$||x - z|| = \left(\frac{1-\lambda}{\lambda}\right) ||x - y|| = \left(\frac{1-\lambda}{\lambda}\right) r = \frac{\delta}{2}$$

So $z \in B(x,\delta) \subset A \subset \operatorname{cl}(A) = \operatorname{cl}(B)$. Now $z \in \operatorname{cl}(B)$ and $y \in B = \operatorname{int}(B)$. So by 2.12, $\operatorname{relint}(\overline{yz}) \subset \operatorname{int}(B) = B$. So $x \in B$ and $A \subset B$. Similarly, $B \subset A$. So $A = B$.

Consider $S = \mathbb{Q} \subset \mathbb{R}$. We have $\operatorname{cl}(S) = \mathbb{R}$ and $\operatorname{int}(\operatorname{cl}(S)) = \mathbb{R}$ while $\operatorname{int}(S) = \emptyset$.

2.15 (p. 23) Since $\operatorname{bd}(\operatorname{cl}(S)) = \operatorname{cl}(\operatorname{cl}(S)) \setminus \operatorname{int}(\operatorname{cl}(S)) = \operatorname{cl}(S) \setminus \operatorname{int}(\operatorname{cl}(S))$, $\operatorname{bd}(S) = \operatorname{cl}(S) \setminus \operatorname{int}(S)$ and $\operatorname{int}(\operatorname{cl}(S)) = \operatorname{int}(S)$ by 2.14, $\operatorname{bd}(\operatorname{cl}(S)) = \operatorname{bd}(S)$.

2.16 (p. 23) Note that $\alpha S + \beta S = \{\alpha x + \beta y : x, y \in S\}$ and $(\alpha+\beta)S = \{(\alpha+\beta)z : z \in S\}$. Please pay attention to the difference between these two sets (where I use different letters x, y, z deliberately).

It is obvious that $(\alpha+\beta)S \subset \alpha S + \beta S$ (letting $x = y = z$). It remains to prove that $\alpha S + \beta S \subset (\alpha+\beta)S$, i.e., $\alpha x + \beta y \in (\alpha+\beta)S$ for all $x, y \in S$.

Let

$$z = \frac{\alpha}{\alpha+\beta}x + \frac{\beta}{\alpha+\beta}y$$

Since

$$\frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta} = 1 \text{ and } \alpha, \beta > 0$$

z is a convex combination of x and y. And since S is convex, $z \in S$ and hence $(\alpha+\beta)z \in (\alpha+\beta)S$. Of course, $(\alpha+\beta)z = \alpha x + \beta y$. So $\alpha x + \beta y \in (\alpha+\beta)S$.

For counterexample that $\alpha S + \beta S \neq (\alpha+\beta)S$ if S is not convex, let $\alpha = \beta = 1$ and $S = \{0,1\} \subset \mathbb{R}$. Then $S + S = \{0+0, 1+1, 0+1\} = \{0,1,2\}$ while $(1+1)S = 2S = \{0,2\}$.

2.21 (p. 23) (a) Since $B \subset \operatorname{conv}(B)$ and $A \subset B$, $A \subset \operatorname{conv}(B)$.

(b) The convex hull of A is the intersection of all convex sets that contain A. Let $\{W_\lambda : \lambda \in I\}$ be the collection of all convex sets that contain A. Then $\operatorname{conv}(A) = \cap_{\lambda\in I} W_\lambda$. Since B is convex and $B \supset A$, $B \in \{W_\lambda : \lambda \in I\}$. So $B = W_\alpha$ for some $\alpha \in I$. Therefore, $\operatorname{conv}(A) = \cap_{\lambda\in I} W_\lambda \subset W_\alpha = B$.

(c) By (a), $A \subset \operatorname{conv}(B)$. By (c), $A \subset \operatorname{conv}(B)$ and $\operatorname{conv}(B)$ is convex $\Rightarrow$ $\operatorname{conv}(A) \subset \operatorname{conv}(B)$.

(d) By (c), $A \subset A \cup B \Rightarrow \operatorname{conv}(A) \subset \operatorname{conv}(A \cup B)$ and $B \subset A \cup B \Rightarrow \operatorname{conv}(B) \subset \operatorname{conv}(A \cup B)$. So $\operatorname{conv}(A) \cup \operatorname{conv}(B) \subset \operatorname{conv}(A \cup B)$.

(e) By (c), $A \cap B \subset A \Rightarrow \operatorname{conv}(A \cap B) \subset \operatorname{conv}(A)$ and $A \cap B \subset B \Rightarrow \operatorname{conv}(A \cap B) \subset \operatorname{conv}(B)$. So $\operatorname{conv}(A \cap B) \subset \operatorname{conv}(A) \cap \operatorname{conv}(B)$.

(f) For a convex set S, $S = \operatorname{conv}(S)$. Since $\operatorname{conv}(A)$ is convex, $\operatorname{conv}(A) = \operatorname{conv}(\operatorname{conv}(A))$.

(g) For (d), let $A = (0, 1)$ and $B = (2, 3) \subset \mathbb{R}$. Then $\operatorname{conv}(A) = A$ and $\operatorname{conv}(B) = B$ since both A and B are convex. But $\operatorname{conv}(A \cup B) = (0, 3) \neq \operatorname{conv}(A) \cup \operatorname{conv}(B) = A \cup B$.

For (e), let $A = \{0, 2\}$ and $B = \{1, 3\} \subset \mathbb{R}$. Then $\operatorname{conv}(A) = (0, 2)$ and $\operatorname{conv}(B) = (1, 3)$. But $\operatorname{conv}(A \cap B) = \operatorname{conv}(\emptyset) = \emptyset \neq \operatorname{conv}(A) \cap \operatorname{conv}(B) = (1, 2)$.

2.24 (p. 23) Note that if $x = \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_m x_m$ and $\alpha_i \geq 0$, then

$$\frac{1}{\alpha} x = \frac{\alpha_1}{\alpha} x_1 + \frac{\alpha_2}{\alpha} x_2 + ... + \frac{\alpha_m}{\alpha} x_m$$

is a convex combination of $x_1, x_2, ..., x_m$, where $\alpha = \alpha_1 + \alpha_2 + ... + \alpha_m$. So

$$\operatorname{pos}(S) = \{\lambda x : x \in \operatorname{conv}(S), \lambda \geq 0\}$$

Geometrically, $\operatorname{pos}(S)$ is the cone over $\operatorname{conv}(S)$ with vertex at the origin. So for $S = \{(-1, 1), (1, 1)\}$, let $P = (-1, 1)$ and $Q = (1, 1)$. Then $\operatorname{conv}(S) = \overline{PQ}$ and $\operatorname{pos}(S)$ is the sector beteween the rays OP and OQ. That is, $\operatorname{pos}(S) = \{y - x \geq 0, y + x \geq 0\}$.

2.27 (p. 24) (a) Suppose that $\{x_1, x_2, ..., x_k\}$ is affinely dependent. Then there exist $\lambda_1, \lambda_2, ..., \lambda_k$, not all zero, such that $\sum_{i=1}^k \lambda_i = 0$ and

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + ... + \lambda_k x_k = 0. \tag{0.1}$$

Then $\lambda_1 = -(\lambda_2 + \lambda_3 + ... + \lambda_k)$. Substitute λ_1 in (0.1):

$$\begin{aligned}&-(\lambda_2 + \lambda_3 + ... + \lambda_k)x_1 + \lambda_2 x_2 + \lambda_3 x_3 + ... + \lambda_k x_k = 0\\ &\Rightarrow \lambda_2(x_2 - x_1) + \lambda_3(x_3 - x_1) + ... + \lambda_k(x_k - x_1) = 0\end{aligned}$$

It is not hard to see that $\lambda_2, \lambda_3, ..., \lambda_k$ cannot be all zero since, otherwise, $\lambda_2 = \lambda_3 = ... = \lambda_k = 0 \Rightarrow \lambda_1 = -(\lambda_2 + \lambda_3 + ... + \lambda_k) = 0$, which is a contradiction. So $x_2 - x_1, x_3 - x_1, ..., x_k - x_1$ are linearly independent.

On the other hand, suppose that $x_2 - x_1, x_3 - x_1, ..., x_k - x_1$ are linearly independent. Then there exist $\lambda_2, \lambda_3, ..., \lambda_k$, not all zero, such that

$$\lambda_2(x_2 - x_1) + \lambda_3(x_3 - x_1) + ... + \lambda_k(x_k - x_1) = 0$$

So

$$-(\lambda_2+\lambda_3+...+\lambda_k)x_1+\lambda_2x_2+\lambda_3x_3+...+\lambda_kx_k=0.$$

Let $\lambda_1=-(\lambda_2+\lambda_3+...+\lambda_k)$. Then $\sum_{i=1}^k\lambda_i=0$, $\sum_{i=1}^k\lambda_ix_i=0$ and λ_i's are not all zero. So $x_1,x_2,...,x_k$ are affinely dependent.

(b) Suppose that $x_1,x_2,...,x_k$ are affinely dependent. Then there exists $\lambda_1,\lambda_2,...,\lambda_k$, not all zero, such that $\sum_{i=1}^k\lambda_i=0$ and $\sum_{i=1}^k\lambda_ix_i=0$. Without the loss of generality, let us assume that $\lambda_1\neq 0$. Then

$$x_1=-\frac{\lambda_2}{\lambda_1}x_2-\frac{\lambda_3}{\lambda_1}x_3-...-\frac{\lambda_k}{\lambda_1}x_k$$

Since

$$-\frac{\lambda_2}{\lambda_1}-\frac{\lambda_3}{\lambda_1}-...-\frac{\lambda_k}{\lambda_1}=-\frac{\lambda_2+\lambda_3+...+\lambda_k}{\lambda_1}=\frac{\lambda_1}{\lambda_1}=1$$

x_1 is an affine combination of $x_2,x_3,...,x_k$.

On the other hand, suppose that x_1 is an affine combination of $x_2,x_3,...,x_k$. Then there exist $\lambda_2,\lambda_3,...,\lambda_k$ such that $\lambda_2+\lambda_3+...+\lambda_k=1$ and $x_1=\lambda_2x_2+\lambda_3x_3+...+\lambda_kx_k$. So

$$-x_1+\lambda_2x_2+\lambda_3x_3+...+\lambda_kx_k=0$$

and

$$(-1)+\lambda_2+\lambda_3+...+\lambda_k=0$$

Therefore, $x_1,x_2,...,x_k$ are affinely dependent.

2.28 (a) Let us assume that p can be written as an affine combination of $x_1,x_2,...,x_{k+1}$ in more than one way. Then

$$p=\lambda_1x_1+\lambda_2x_2+...+\lambda_{k+1}x_{k+1}=\lambda_1'x_1+\lambda_2'x_2+...+\lambda_{k+1}'x_{k+1}$$

for some $\sum\lambda_i=\sum\lambda_i'=1$ and

$$(\lambda_1,\lambda_2,...,\lambda_{k+1})\neq(\lambda_1',\lambda_2',...,\lambda_{k+1}')$$

Then

$$(\lambda_1-\lambda_1')x_1+(\lambda_2-\lambda_2')x_2+...+(\lambda_{k+1}-\lambda_{k+1}')x_{k+1}=0$$

with $\sum(\lambda_i-\lambda_i')=\sum\lambda_i-\sum\lambda_i'=0$.

Moreover, since $(\lambda_1,\lambda_2,...,\lambda_{k+1})\neq(\lambda_1',\lambda_2',...,\lambda_{k+1}')$, $\lambda_i-\lambda_i'$ are not all zero. Therefore, $x_1,x_2,...,x_{k+1}$ are affinely dependent. By 2.27 (b), one of $x_1,x_2,...,x_{k+1}$ is an affine combination of the rest. Without the loss of generality, let us assume that $x_{k+1}\in\operatorname{aff}\{x_1,x_2,...,x_k\}$. Then $\operatorname{aff}(S)=\operatorname{aff}\{x_1,x_2,...,x_k\}$ has dimension less than k. This is a contradiction.

(b) To find the affine coordinates of a point, it comes to down to solve a system of linear equations. The answers are $p_1=(12/13,3/13,-2/13)$, $p_2=(8/13,2/13,3/13)$, $p_3=(2/3,0,1/3)$ and $p_4=(9/13,-1/13,5/13)$.

(c) It is in the interior iff all the affine coordinates are positive. It is on the boundary if all the affine coordinates are nonnegative and one of them is zero. So p_2 is in the interior, p_3 is on the boundary and p_1 and p_4 are outside.

(d) By the criterion given in (c), y_4 is in the interior, y_2 is on the boundary and y_1 and y_3 are outside.

2.29 A flat is a translate of a linear subspace. Let $F = V + x$ and $G = W + y$, where V and W are linear subspaces of dimension k. Since $F \subset G$, $V + x \subset W + y$. So $V \subset W + (y - x)$. Since $0 \in V$, $0 \in W + (y - x)$. Hence $x - y \in W \Rightarrow y - x \in W$. Therefore, $W + (y - x) = W$ and $V \subset W$. And since $\dim W = \dim V$, we must have $V = W$. This implies $V = W + (y - x)$ and $V + x = W + y$. That is, $F = G$.