Math 341 Homework 2 Solution

2.6 (p. 22) Let $x_1 = a_1 + b_1$ and $x_2 = a_2 + b_2$ be two points in $A+B$ with $a_1, a_2 \in A$ and $b_1, b_2 \in B$. We want to show that

$$\overline{x_1x_2} = \{\alpha x_1 + \beta x_2 : \alpha, \beta \geq 0, \alpha + \beta = 1\} \subset A + B$$

Since

$$\alpha x_1 + \beta x_2 = \alpha(a_1 + b_1) + \beta(a_2 + b_2) = (\alpha a_1 + \beta a_2) + (\alpha b_1 + \beta b_2)$$

$\alpha a_1 + \beta a_2 \in A$ because A is convex and $\alpha b_1 + \beta b_2 \in B$ because B is convex, consequently, $\alpha x_1 + \beta x_2 \in A + B$ for any $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. So $A + B$ is convex.

2.7 (p. 22) (a) Let $C = \{y = x^2\}$. Since $f(x) = x^2$ is concave upward, $\text{conv}(C) = \{y \geq x^2\}$.

(b) Let $C = \{y = x^2, x \geq 0\}$ and $W = \{y \geq x^2, x > 0\} \cup \{(0,0)\}$. We want to show that $W = \text{conv}(C)$.

Since both $\{y \geq x^2\}$ and $\{x > 0\} \cup \{(0,0)\}$ are convex, W is the intersection of these two sets and hence convex. Let $O = (0,0)$ and $P_x = (x, x^2) \in C$ for each $x \geq 0$. Since $\overline{OP_x} \subset \text{conv}(C)$, $\cup_{x\geq 0}\overline{OP_x} \subset \text{conv}(C)$. Obviously,

$$\bigcup_{x\geq 0} \overline{OP_x} = W$$

since the slope of the line OP_x is x which approaches ∞ as $x \to \infty$. Therefore, $C \subset W \subset \text{conv}(C)$. And since W is convex, $\text{conv}(C) = W$.

(c) Let $C = \{y = x^3\}$. We want to show that $\text{conv}(C) = \mathbb{R}^2$.

Let $C_+ = \{y = x^3, x > 0\}$ and $W_+ = \text{conv}(C_+)$. Since x^3 is concave upward when $x > 0$, using the same argument as in (b), we see that $W_+ = \{y \geq x^3, x > 0\}$.

Let $C_- = \{y = x^3, x < 0\}$ and $W_- = \text{conv}(C_-)$. By the same argument as before, we see that $W_- = \text{conv}(C_-) = \{y \leq x^3, x < 0\}$. Therefore, $W_+ \cup W_- \subset \text{conv}(C)$ and $\text{conv}(W_+ \cup W_-) \subset \text{conv}(C)$. Next we will show that $\mathbb{R}^2 \subset \text{conv}(W_+ \cup W_-)$. Let $P = (x_p, y_p)$ be a point not in $W_+ \cup W_-$. Without the loss of generality, let us assume that $x_p \leq 0$ (the argument for $x_p \geq 0$ proceeds in the same way). Since $P \notin W_-$, $y_p \geq x_p^3$. Choose a point $Q = (x_q, x_q^3) \in C_-$ with $x_q < x_p$. The slope of PQ is

$$\frac{y_p - x_q^3}{x_p - x_q} \geq \frac{x_p^3 - x_q^3}{x_p - x_q} > 0$$

i.e., the line PQ has positive slope. Hence it will meet W_+ at some point R with $P \in \overline{QR}$. Therefore, $P \in \text{conv}(W_+ \cup W_-)$ and $\mathbb{R}^2 \subset \text{conv}(W_+ \cup W_-)$. So $\mathbb{R}^2 = \text{conv}(C)$.

(d) Let $C = \{y = 1/x, x \geq 1/2\}$ and $W = \{y \geq 1/x, y < 2, x > 0\} \cup \{(1/2, 2)\}$. We want to show that $W = \text{conv}(C)$.

Since $y = 1/x$ is concaved upward when $x > 0$, $\{y \geq 1/x, x > 0\}$ is convex. And since $\{y < 2\} \cup \{(1/2, 2)\}$ is convex, W is convex.

Let $P = (1/2, 2)$. For each $x > 1/2$, let $Q_x = (x, 1/x) \in C$. Then $\overline{PQ_x} \subset \text{conv}(C)$ for every $x > 1/2$. Since the slope of the line PQ_x is $-1/(2x)$ and $\lim_{x\to\infty} -1/(2x) = 0$,

$$W = \bigcup_{x>1/2} \overline{PQ_x}.$$

So $W \subset \text{conv}(C)$. On the other hand, $C \subset W$ and W is convex.

(e) Let $C = \{y = \sin x\}$ and $W = \{-1 \le y \le 1\}$. We want to show that $W = \text{conv}(C)$.

It is obvious that W is convex.

Fix θ. For each positive integer n, let $P_n = (\theta - 2n\pi, \sin\theta)$ and $Q_n = (\theta + 2n\pi, \sin\theta)$. Both P_n and Q_n lie on C. So $\overline{P_nQ_n} \subset \text{conv}(C)$. Obviously,

$$\bigcup_{n=1}^{\infty} \overline{P_nQ_n} = L_\theta$$

where L_θ is the horizontal line passing through P_n and Q_n, i.e., $L_\theta = \{y = \sin\theta\}$. So $L_\theta \subset \text{conv}(C)$ for every θ. Therefore, $W = \cup L_\theta$ is contained in $\text{conv}(C)$. On the other hand, $C \subset W$ and W is convex. Hence $W = \text{conv}(C)$.

(f) Let $C = \{y = \tan^{-1} x, -\pi/2 < y < \pi/2\}$ and $W = \{-\pi/2 < y < \pi/2\}$. It follows from a similar argument as in (c) that $W = \text{conv}(C)$.

Let $C_+ = \{y = \tan^{-1} x, x > 0\}$ and $C_- = \{y = \tan^{-1} x, x < 0\}$. Then $W_+ = \text{conv}(C_+) = \{0 < y \le \tan^{-1} x\}$ and $W_- = \text{conv}(C_-) = \{0 > y \ge \tan^{-1} x\}$. Next using a similar argument as in (c), one can show that $W \subset \text{conv}(W_+ \cup W_-)$.

2.8 (p. 22) Let $y, z \in B(x, \delta)$, i.e., $||y - x|| < \delta$ and $||z - x|| < \delta$. We want to show $\alpha y + \beta z \in B(x, \delta)$ for every $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$. Since

$$\begin{aligned} ||\alpha y + \beta z - x|| &= ||\alpha(y - x) + \beta(z - x)|| \le ||\alpha(y - x)|| + ||\beta(z - x)|| \\ &= \alpha||y - x|| + \beta||z - x|| < \alpha\delta + \beta\delta = \delta \end{aligned}$$

$\alpha y + \beta z \in B(x, \delta)$. So $B(x, \delta)$ is convex.

2.9 (p. 22) Let S be a bounded set. Then $S \subset B(x, R)$ for some point x and $R > 0$. Hence $\text{conv}(S) \subset \text{conv}(B(x, R))$. Since $B(x, R)$ is convex, $\text{conv}(B(x, R)) = B(x, R)$. Therefore, $\text{conv}(S) \subset B(x, R)$ and $\text{conv}(S)$ is bounded.

2.10 (p. 22) A flat is a translate of linear subspace of $\mathbb{R}^n$. Let $S = W + x$ be a flat with W a linear subspace of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$. Let $y + x \in S$ and $z + x \in S$. Then

$$\alpha(y + x) + \beta(z + x) = \alpha y + \beta z + x$$

for any $\alpha, \beta \ge 0$ and $\alpha + \beta = 1$. Since W is a linear subspace and $y, z \in W$, $\alpha y + \beta z \in W$. Hence $\alpha(y + x) + \beta(z + x) \in S$ and S is convex.

2.11 (p. 22) Let $W = \overline{xy} \cap S$. Since both $\overline{xy}$ and S are closed, W is closed. Since $\overline{xy}$ is bounded, $W \subset \overline{xy}$ is bounded. Let $f : W \to \mathbb{R}$ be the function $f(w) = d(x, w)$ for $w \in W$. We have proved that such f is continuous. And since W is compact, $f(w)$ has a maximum and minimum over W (Extreme

Value Theorem: a continuous function has a maximum and minimum over a compact set). Suppose that f achieves the minimum at $z \in W$. We want to show that $z \in \mathrm{bd}(S)$. If not, then $z \in \mathrm{int}(S)$. There exists an open ball $B(z,\delta) \subset S$ with radius $\delta > 0$. Let $r = d(x,z) = f(z)$ be the distance between x and z. Then there exists a point $p \in \overline{xz}$ such that $d(x,p) = r - \delta/2$ and $d(p,z) = \delta/2$. So $p \in B(z,\delta) \subset S$ and $p \in \overline{xy}$. Consequently, $p \in W$. But $f(p) = d(x,p) = r - \delta/2 < f(z)$. This contradicts the fact that f has a minimum at z. So $z \in \mathrm{bd}(S)$.

2.12 (p. 22) Since $y \in \mathrm{int}(S)$, there exists an open ball $B(y,r) \subset S$ with $r > 0$. Obviously, $B(y,r) = \mathrm{int}(B(y,r)) \subset \mathrm{int}(S)$, i.e., every point of $B(y,r)$ is an interior point of S.

Since $x \in \mathrm{cl}(S)$, for every $\delta > 0$, there exists a point $z \in S$ such that $||x - z|| < \delta$.

Fix $\delta > 0$. We claim that

$$w = (1-\lambda)x + \lambda y \in \mathrm{int}(S)$$

for every $\delta/r < \lambda < 1$. Then $\mathrm{relint}(\overline{xy}) \subset \mathrm{int}(S)$ follows easily by letting $\delta \to 0$.

Let $z \in S$ be the point such that $||x - z|| < \delta$. Let

$$u = \frac{1}{\lambda}(w - (1-\lambda)z)$$

Since $w = (1-\lambda)z + \lambda u$, $w \in \overline{uz}$ and $w \neq u, z$ because $\lambda \neq 0, 1$, i.e., $w \in \mathrm{relint}(\overline{uz})$.

Since

$$\begin{aligned} ||u - y|| &= \frac{1}{\lambda}||w - (1-\lambda)z - \lambda y|| \\ &= \frac{1}{\lambda}||(1-\lambda)x + \lambda y - (1-\lambda)z - \lambda y|| \\ &= \frac{1-\lambda}{\lambda}||x - z|| < (1-\lambda)\frac{\delta}{\lambda} < (1-\lambda)r < r \end{aligned}$$

$u \in B(y,r)$ and hence $u \in \mathrm{int}(S)$. Since $z \in S$ and $u \in \mathrm{int}(S)$, $\mathrm{relint}(\overline{uz}) \subset \mathrm{int}(S)$. Therefore, $w \in \mathrm{int}(S)$.

So for every $\delta > 0$ and $\delta/r < \lambda < 1$, $(1-\lambda)x + \lambda y \in \mathrm{int}(S)$. By letting $\delta \to 0$, we see that $(1-\lambda)x + \lambda y \in \mathrm{int}(S)$ for every $0 < \lambda < 1$, i.e., $\mathrm{relint}(\overline{xy}) \subset \mathrm{int}(S)$.

2.19 (p. 23) (a) Consider $S = (0,1)$. S is convex but $\mathrm{bd}(S) = \{0,1\}$ is not.

(b) Consider $S = (0,\infty)$. S is convex and $\mathrm{bd}(S) = \{0\}$ is convex, too.

(c) Pick a point $x \in \mathrm{int}(S)$. Let L be a line passing through x. Set $W = L \cap \mathrm{cl}(S)$. Since both L and $\mathrm{cl}(S)$ are convex, W is convex. Since both L and $\mathrm{cl}(S)$ are closed, W is closed. Since $\mathrm{cl}(S)$ is bounded, W is bounded. So W is a closed bounded convex subset of $L \cong \mathbb{R}^1$; a closed bounded convex subset of $\mathbb{R}^1$ must be a closed interval $[a,b]$. Therefore, $W = \overline{pq}$ with $p, q \in L$. We claim that $p, q \in \mathrm{bd}(S)$. If not, say $p \notin \mathrm{bd}(S)$. Then

$p \in \text{int}(S)$. There exists an open ball $B(p,\delta) \subset S$ with $\delta > 0$. Obviously, $L \cap B(p,\delta) \subset W$ but $L \cap B(p,\delta) \not\subset \overline{pq}$. Contradiction. So $p, q \in \text{bd}(S)$. Since $x \in \overline{pq}$ and $x \notin \text{bd}(S)$, $\text{bd}(S)$ is not convex.

A1. First we show that $W = \{y \geq f(x)\}$ is convex. Let $p, q \in W$. We want to show that $\overline{pq} \subset W$. If not, suppose that there exists a point $r \in \overline{pq}$ and $r \notin W$.

Let $p = (x_p, y_p), q = (x_q, y_q), r = (x_r, y_r)$. Then $x_p < x_r < x_q$. By Mean Value Theorem (MVT), there exists $a \in (x_p, x_r)$ such that

$$f'(a) = \frac{f(x_r) - f(x_p)}{x_r - x_p}$$

Again by MVT, there exists $b \in (x_r, x_q)$ such that

$$f'(b) = \frac{f(x_q) - f(x_r)}{x_q - x_r}$$

Since $p, q \in W$, $y_p \geq f(x_p)$ and $y_q \geq f(x_q)$ and since $r \notin W$, $y_r < f(x_r)$. Therefore,

$$f'(a) = \frac{f(x_r) - f(x_p)}{x_r - x_p} > \frac{y_r - y_p}{x_r - x_p} = m_{pr}$$

and

$$f'(b) = \frac{f(x_q) - f(x_r)}{x_q - x_r} < \frac{y_q - y_r}{x_q - x_r} = m_{rq}$$

where m_{pr} and m_{rq} are the slopes of the lines pr and rq, respectively. Of course, $m_{pr} = m_{rq}$ as p, q, r lie on the same line. So $f'(a) > f'(b)$ with $a < b$. This is impossible because $f''(x) > 0$ and $f'(x)$ is increasing. Therefore, $\overline{pq} \subset W$ and W is convex.

Next we show that $W \subset \text{conv}(C)$, where $C = \{y = f(x)\}$. Let $r = (x_r, y_r) \in W$ and $r \notin C$. We will show that there exist two points $p, q \in C$ such that $r \in \overline{pq}$.

Let $m_0 = f'(0)$, $m_1 = f'(1)$ and $m_2 = f'(2)$ (there is nothing special about $0, 1, 2$ and you may choose three arbitrary numbers $x_0 < x_1 < x_2$ and let $m_0 = f'(x_0)$, $m_1 = f'(x_1)$ and $m_2 = f'(x_2)$ be the derivatives). Obviously, $m_0 < m_1 < m_2$.

Let L_r be the line passing through point r with slope m_1, i.e.,

$$L_r : y - y_r = m_1(x - x_r)$$

We want to show that L_r meets the curve C at two points p and q. Let

$$g(x) = f(x) - (y_r + m_1(x - x_r))$$

Since $r \in W$ and $r \notin C$, $y_r > f(x_r)$ and hence $g(x_r) < 0$. When $x > 2$,

$$f(x) = f(2) + \int_2^x f'(t)dt > f(2) + m_2(x - 2)$$

Therefore, $\lim_{x\to\infty} g(x) = \infty$. When $x < 0$,

$$f(x) = f(2) - \int_x^0 f'(t)dt > f(2) - m_0(0 - x)$$

Therefore, $\lim_{x\to-\infty} g(x) = \infty$.

Since $g(x_r) < 0$ and $\lim_{x\to\infty} g(x) = \lim_{x\to-\infty} g(x) = \infty$, by Intermediate Value Theorem, there exist $x_p < x_r$ and $x_q > x_r$ such that $f(x_p) = f(x_q) = 0$. Correspondingly, $p = (x_p, f(x_p))$ and $q = (x_q, f(x_q))$ are the intersections between L_r and C. Hence $r \in \overline{pq}$ and $W \subset \text{conv}(C)$.

Since $C \subset W \subset \text{conv}(C)$ and W is convex, $W = \text{conv}(C)$.

A2. Let $C = \{y = f(x)\}$ and $C' = \{y = -f(x)\}$. Since $-f(x)$ is concave upward, by the previous problem, $\text{conv}(C') = \{y \geq -f(x)\}$. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the map $\phi(x, y) = (x, -y)$. Then $C = \phi(C')$ and $\phi(\text{conv}(C')) = \text{conv}(\phi(C'))$ because ϕ maps convex sets to convex sets. So

$$\text{conv}(C) = \phi(\{y \geq -f(x)\}) = \{-y \geq -f(x)\} = \{y \leq f(x)\}$$