Math 341 Homework 1 Solution

**1.3 (p. 9)** (a) Since

$$\begin{aligned}(A+B)+C &= \{a+b : a\in A, b\in B\} + \{c : c\in C\}\\ &= \{a+b+c : a\in A, b\in B, c\in C\}\end{aligned}$$

and

$$\begin{aligned}A+(B+C) &= \{a : a\in A\} + \{b+c : b\in B, c\in C\}\\ &= \{a+b+c : a\in A, b\in B, c\in C\}\end{aligned}$$

it follows that $(A+B)+C = A+(B+C)$.

(b) Since

$$\alpha(A+B) = \alpha\{a+b : a\in A, b\in B\} = \{\alpha a+\alpha b : a\in A, b\in B\}$$

and

$$\alpha A+\alpha B = \{\alpha a : a\in A\} + \{\alpha b : b\in B\} = \{\alpha a+\alpha b : a\in A, b\in B\}$$

it follows that $\alpha(A+B) = \alpha A+\alpha B$.

**1.5 (p. 9)** (a) Let $S = B(p,\delta)$ be the open ball centered at point $p$ with radius $\delta$. We want to show that $S$ is open. Let $q\in S$. Then $\varepsilon = d(p,q) < \delta$. We claim $B(q,\delta-\varepsilon)\subset S$. Let $r\in B(q,\delta-\varepsilon)$. Then $d(q,r) < \delta-\varepsilon$. By the triangle inequality,

$$d(p,r) \le d(p,q)+d(q,r) < \varepsilon+(\delta-\varepsilon) = \delta$$

Therefore, $r\in B(p,\delta) = S$ and hence $B(q,\delta-\varepsilon)\subset S$. Consequently, $q$ is an interior point of $S$ and $S$ is open.

(b) Every point of $\mathbb{R}^n$ is obviously an interior point of $\mathbb{R}^n$. So $\mathbb{R}^n$ is open.

(c) Since $\mathrm{int}(\emptyset) = \emptyset$, $\emptyset$ is open.

(d) Let $\{V_\lambda : \lambda\in I\}$ be a collection of open sets. We want to show that the union $V = \cup_{\lambda\in I}V_\lambda$ is open. Let $p\in V$ be a point in V. Then $p\in V_\lambda$ for some $\lambda\in I$. Since $V_\lambda$ is open, there exists a $\delta > 0$ such that $B(p,\delta)\subset V_\lambda$ and hence $B(p,\delta)\subset V$. So $V$ is open.

(e) Let $V = \cap_{i=1}^n V_i$ be the intersection of $n$ open sets $V_1, V_2, ..., V_n$. We want to show that $V$ is open. Let $p\in V$. Since $p\in V_i$, there exists $\delta_i > 0$ such that $B(p,\delta_i)\subset V_i$ for each $i$. Let $\delta = \min(\delta_1,\delta_2,...,\delta_n)$. Then $B(p,\delta) = \cap_{i=1}^n B(p,\delta_i)\subset\cap_{i=1}^n V_i = V$. So $p$ is an interior point of $V$ and $V$ is open.

**1.6 (p. 9)** (a) Let $V = \{p_1, p_2, ..., p_n\}$ be a finite set. We want to show that $V$ is closed, i.e., the complement $V^c$ is open. Let $p\in V^c$ and let $r_i = d(p,p_i)$. Since $p\notin V$, $r_i > 0$ for each $i$. Let $r = \min(r_1, r_2, ..., r_n)$. Then $B(p,r)\cap V = \emptyset$ and hence $B(p,r)\subset V^c$. So $V^c$ is open and $V$ is closed.

(b) The complement of $\mathbb{R}^n$ is $\emptyset$, which is open. So $\mathbb{R}^n$ is closed.

(c) The complement of $\emptyset$ is $\mathbb{R}^n$, which is open. So $\emptyset$ is closed.

(d) Let $\{V_\lambda : \lambda \in I\}$ be a collection of closed sets. We want to show that $V = \cap_{\lambda\in I} V_\lambda$ is closed. Since $V^c = \cup_{\lambda\in I} V_\lambda^c$ and $V_\lambda^c$ is open for each $\lambda$, $V^c$ is open and hence $V$ is closed.

(e) Let $V = \cup_{i=1}^n V_i$ be the union of $n$ closed sets $V_i$. Since $V^c = \cap_{i=1}^n V_i^c$ and $V_i^c$ is open for each $i$, $V^c$ is open and hence $V$ is closed.

**1.7 (p. 9)** (a) For every point $p \in \text{int}(A)$, there exists an open ball $B(p,r) \subset A$ for some $r > 0$. Since $A \subset B$, $B(p,r) \subset B$ and hence $p \in \text{int}(B)$. Therefore, $\text{int}(A) \subset \text{int}(B)$.

(b) The closure of $A$ is the intersection $\cap_{i\in I} V_i$, where $\{V_i : i \in I\}$ is the collection of all closed sets that contain $A$. Similarly, the closure of $B$ is the intersection $\cap_{j\in J} U_j$, where $\{U_j : j \in J\}$ is the collection of all closed sets that contain $B$. Since $A \subset B$, every closed set that contains $B$ also contains $A$, i.e., $\{U_j : j \in J\} \subset \{V_i : i \in I\}$. Therefore, $\cap_{i\in I} V_i \subset \cap_{j\in J} U_j$, i.e., $\text{cl}(A) \subset \text{cl}(B)$.

**1.10 (p. 10)** (a) Since $A + B = \cup_{b\in B}(A + b)$ and $A + b \cong A$ is open, $A + B$ is open since the union of a collection of open sets is open.

(b) This is false. Use the example I gave in the hint. Let $A = \{1, 2, ..., n, ...\}$ and $B = \{1/2-1, 1/4-2, ..., 2^{-n}-n, ...\}$. So $A+B = \{m-n+2^{-n} : m, n \in \mathbb{Z}, m, n > 0\}$. We want to show that both $A$ and $B$ are closed, while $A + B$ is not. Since $A^c = (-\infty, 1) \cup (1, 2) \cup (2, 3) \cup ... \cup (n, n+1) \cup ...$ is the union of open intervals, $A^c$ is open and hence $A$ is closed. Similarly, $B$ is closed since $B^c = (1/2 - 1, \infty) \cup (1/4 - 2, 1/2 - 1) \cup ... \cup (2^{-n-1} - n - 1, 2^{-n} - n) \cup ...$ is open. To show that $A + B$ is not closed, we first show that $0 \notin A + B$. This is obvious since $m-n+2^{-n}$ is not an integer for any $m, n \in \mathbb{Z}$ and $m, n > 0$. Next we show that for any $\varepsilon > 0$, $(-\varepsilon, \varepsilon) \cap (A + B) \neq \emptyset$. Since $\varepsilon > 0$, there exists $N \in \mathbb{Z}$ and $N > 0$ such that $2^{-N} < \varepsilon$. And since $2^{-N} \in (A + B)$, $2^{-N} \in (-\varepsilon, \varepsilon) \cap (A + B) \neq \emptyset$. Therefore, $0 \in (A + B)^c$ and 0 is not interior point of $(A + B)^c$ because $(-\varepsilon, \varepsilon) \not\subset (A + B)^c$. So $(A + B)^c$ is not open and $A + B$ is not closed.

**1.13 (p. 10)** To show that a function is continuous, it is more convenient to use the $\epsilon - \delta$ definition than the standard definition. In this case, we want to show that $f(x)$ is continous at every point $x \in \mathbb{R}^n$ for each $\epsilon > 0$, there exists $\delta > 0$ such that $f(B(x, \delta)) \subset B(f(x), \epsilon)$. For this function $f(x) = d(x, p)$, it is enough to pick $\delta = \epsilon$. For each $y \in B(x, \delta)$, i.e., $y$ satisfying $d(x, y) < \delta$,

$$|f(y) - f(x)| = |d(y, p) - d(x, p)| \leq d(x, y) < \delta = \epsilon$$

Therefore, $f(y) \in B(f(x), \epsilon)$ and hence $f(B(x, \delta)) \subset B(f(x), \epsilon)$. So $f(x)$ is continuous at every point $x \in \mathbb{R}^n$.

Another way to see $f(x)$ is continuous is to write down the formula for $f(x)$. Let $p = (p_1, p_2, ..., p_n)$. Keep in mind that $p_1, p_2, ..., p_n$ are constants as $p$ is fixed. Let $x = (x_1, x_2, ..., x_n)$. Then

$$f(x) = f(x_1, x_2, ..., x_n) = \sqrt{(x_1 - p_1)^2 + (x_2 - p_2)^2 + ... + (x_n - p_n)^2}$$

You may take the following facts for granted (although they are not that obvious): the sum, difference, product, quotient (as long as the denominator is not zero) of two continuous functions are continuous; the composition of two continuous functions are continuous.

Now $x_1 - p_1, x_2 - p_2, ..., x_n - p_n$ are all continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$. So $(x_1 - p_1)^2, (x_2 - p_2)^2, ..., (x_n - p_n)^2$ are continuous and hence $g(x_1, x_2, ..., x_n) = (x_1 - p_1)^2 + (x_2 - p_2)^2 + ... + (x_n - p_n)^2$ is continuous. Now $f$ is the composition $h \circ g$ with $h(z) = \sqrt{z}$. Since both $h$ and $g$ are continuous, $f$ is continuous.

**1.14 (p. 10)** Let $r = ||p||$. Take $\delta = \epsilon/r$. For every $y \in B(x, \delta)$, i.e., $y$ satisfying $||y - x|| < \delta$,

$$\begin{aligned}|f(y) - f(x)| &= |\langle y, p\rangle - \langle x, p\rangle| = |\langle y - x, p\rangle| \leq ||y - x|| \cdot ||p|| \\ &< \delta r = (\epsilon/r)r = \epsilon\end{aligned}$$

Therefore, $f(y) \in B(f(x), \epsilon)$ and hence $f(B(x, \delta)) \subset B(f(x), \epsilon)$. So $f(x)$ is continuous.

Here is another way to show it. Let $p = (p_1, p_2, ..., p_n)$. Then

$$f(x) = f(x_1, x_2, ..., x_n) = p_1x_1 + p_2x_2 + ... + p_nx_n$$

Since $p_1x_1, p_2x_2, ..., p_nx_n$ are continuous functions, $p_1x_1 + p_2x_2 + ... + p_nx_n$ is continuous.

**1.22 (p. 10)** Suppose that $f(A)$ is not connected. Then $f(A) = Y \cup Z$ with $Y, Z \neq \emptyset$ and $Y \cap \mathrm{cl}(Z) = \mathrm{cl}(Y) \cap Z = \emptyset$. Then

$$A \subset f^{-1}(Y) \cup f^{-1}(Z).$$

Since $Y \cap Z = \emptyset$, $f^{-1}(Y) \cap f^{-1}(Z) = \emptyset$.

Let $B = A \cap f^{-1}(Y)$ and $C = A \cap f^{-1}(Z)$. Then

$$A = B \cup C$$

Next we want to show that $B$ and $C$ are separated, i.e., $B \cap \mathrm{cl}(C) = \mathrm{cl}(B) \cap C = \emptyset$.

Since $f$ is continuous, the inverse images of closed sets are closed. Hence $f^{-1}(\mathrm{cl}(Z))$ is closed. Since $C \subset f^{-1}(Z) \subset f^{-1}(\mathrm{cl}(Z))$, $\mathrm{cl}(C) \subset f^{-1}(\mathrm{cl}(Z))$.

Since $Y \cap \mathrm{cl}(Z) = \emptyset$, $f^{-1}(Y) \cap f^{-1}(\mathrm{cl}(Z)) = \emptyset$. Consequently, $B \cap f^{-1}(\mathrm{cl}(Z)) = \emptyset$. So $B \cap \mathrm{cl}(C) = \emptyset$. Similarly, $\mathrm{cl}(B) \cap C = \emptyset$. To conclude our proof, we have to show that $B$ and $C$ are not empty.

Since $f(A) = Y \cup Z$ with $Y \cap Z = \emptyset$ and $Y, Z \neq \emptyset$, there exist $y, z \in A$ such that $f(y) \in Y$ and $f(z) \in Z$. Then $y \in B$ and $z \in C$ so $B, C \neq \emptyset$.

Therefore, $B$ and $C$ are separated and $B, C \neq \emptyset$. Hence $A = B \cup C$ is not connected. Contradiction.

**2.2 (p. 10)** (a) Let $x \in A \cap B$ and $y \in A \cup B$. If $y \in A$, then $\overline{xy} \subset A$ because $A$ is convex. If $y \in B$, then $\overline{xy} \subset B$ because $B$ is convex. Therefore, $\overline{xy} \subset A \cup B$. Therefore, $x$ lies in the kernel of $A \cup B$ and $A \cap B$ is contained in the kernel of $A \cup B$.

(b) Consider $A = (0, 2)$ and $B = (1, 3)$. Then $A \cup B = (0, 3)$ is convex and hence the kernel of $A \cup B$ is $A \cup B$ itself. Obviously, $A \cap B \neq A \cup B$ is not the kernel of $A \cup B$.

**A1.**

$$\begin{aligned} ||\vec{x}+\vec{y}||^2 &= \langle \vec{x}+\vec{y}, \vec{x}+\vec{y}\rangle \\ &= \langle \vec{x}, \vec{x}+\vec{y}\rangle + \langle \vec{y}, \vec{x}+\vec{y}\rangle \\ &= \langle \vec{x}, \vec{x}\rangle + \langle \vec{x}, \vec{y}\rangle + \langle \vec{y}, \vec{x}\rangle + \langle \vec{y}, \vec{y}\rangle \\ &= ||\vec{x}||^2 + 2\langle \vec{x}, \vec{y}\rangle + ||\vec{y}||^2 \end{aligned}$$

**A2.** Following the hint, we let

$$f(\lambda) = \langle \vec{x}+\lambda\vec{y}, \vec{x}+\lambda\vec{y}\rangle = ||\vec{x}||^2 + 2\lambda\langle \vec{x}, \vec{y}\rangle + \lambda^2||\vec{y}||^2$$

Since $f(\lambda) \geq 0$ for all $\lambda \geq 0$, the discriminant of $f(\lambda)$ as a quadratic polynomial of $\lambda$ must be $\leq 0$, i.e.,

$$4\langle \vec{x}, \vec{y}\rangle^2 - 4||\vec{x}||^2||\vec{y}||^2 \leq 0$$

This is

$$|\langle \vec{x}, \vec{y}\rangle| \leq ||\vec{x}|| \cdot ||\vec{y}||$$

**A3.** (a) Let $\vec{x} = (x_1, x_2)$. Then

$$\begin{aligned} \langle \vec{x}, \vec{x}\rangle &= 2x_1^2 + 2x_2^2 + 2x_1x_2 \\ &= x_1^2 + x_2^2 + (x_1+x_2)^2 \geq 0 \end{aligned}$$

If $\langle \vec{x}, \vec{x}\rangle = 0$, then $x_1 = x_2 = x_1 + x_2 = 0$, i.e., $\vec{x} = \vec{0}$.

(b) Let $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$. Then

$$\langle \vec{x}, \vec{y}\rangle = 2x_1y_1 + 2x_2y_2 + x_1y_2 + x_2y_1$$

and

$$\langle \vec{y}, \vec{x}\rangle = 2y_1x_1 + 2y_2x_2 + y_1x_2 + y_2x_1$$

So $\langle \vec{x}, \vec{y}\rangle = \langle \vec{y}, \vec{x}\rangle$.

(c) Let $\vec{x} = (x_1, x_2)$, $\vec{y} = (y_1, y_2)$ and $\vec{z} = (z_1, z_2)$. Then

$$\begin{aligned} \langle \vec{x}, \vec{y}+\vec{z}\rangle &= 2x_1(y_1+z_1) + 2x_2(y_2+z_2) + x_1(y_2+z_2) + x_2(y_1+z_1) \\ &= (2x_1y_1 + 2x_2y_2 + x_1y_2 + x_2y_1) + (2x_1z_1 + 2x_2z_2 + x_1z_2 + x_2z_1) \\ &= \langle \vec{x}, \vec{y}\rangle + \langle \vec{x}, \vec{z}\rangle \end{aligned}$$

(d) Let $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$. Then

$$\begin{aligned} \langle \alpha\vec{x}, \vec{y}\rangle &= 2\alpha x_1y_1 + 2\alpha x_2y_2 + \alpha x_1y_2 + \alpha x_2y_1 \\ &= \alpha(2x_1y_1 + 2x_2y_2 + x_1y_2 + x_2y_1) = \alpha\langle \vec{x}, \vec{y}\rangle \end{aligned}$$