Math 341 Homework 1

An important aspect of this course is learning to write mathematical proofs. Most problems below are quite trivial once you understand what is going on. Indeed, we won't do anything really interesting until we get to Helly's theorem. So the first part of the course will be spent on understanding some basic concepts of the subject, such as the concepts of convex sets, vector spaces and so on. It is essential that you be able to write clear and logically sound proofs for some simple mathematical facts.

Homework 1 (due Jan. 23)

Problems from the book: (pp. 9-10) 1.3, 1.5, 1.6, 1.7, 1.10, 1.13, 1.14, 1.22, (p. 22) 2.1, 2.2

Some additional problems:

Some notes on notations: I use $\mathbb{R}^n$ for the n-dimensional real vector space (n-dimensional Euclidean space) and the book uses $\mathbf{E}^n$. I will sometimes (but not always) use the vector notation $\vec{x}$ to denote a point in $\mathbb{R}^n$ to emphasize the fact that it is given by n coordinates $(x_1, x_2, ..., x_n)$ instead of just a number. The book oddly uses θ for the zero vector $(0, 0, ..., 0)$ or the origin. I will just use 0 or $\vec{0}$ for it.

A1. Let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathbb{R}^n$. Show that

$$||\vec{x} + \vec{y}||^2 = ||\vec{x}||^2 + 2\langle \vec{x}, \vec{y} \rangle + ||\vec{y}||^2$$

A2. Show that

$$\langle \vec{x}, \vec{y} \rangle \leq ||\vec{x}||||\vec{y}||$$

for all $\vec{x}, \vec{y} \in \mathbb{R}^n$.

A3. We may define inner products in a nonstandard way. For example, for $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2) \in \mathbb{R}^2$, we define an inner product on $\mathbb{R}^2$ as

$$\langle \vec{x}, \vec{y} \rangle = 2x_1y_1 + 2x_2y_2 + x_1y_2 + x_2y_1$$

Show that this inner product still possesses all the properties in Theorem 1.1 of the book. That is, show the following holds for this inner product:

(a) $\langle \vec{x}, \vec{x} \rangle \geq 0$ and $\langle \vec{x}, \vec{x} \rangle = 0$ iff $\vec{x} = 0$;
(b) $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$;
(c) $\langle \vec{x} + \vec{y}, \vec{z} \rangle = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$;
(d) $\langle \alpha\vec{x}, \vec{y} \rangle = \alpha\langle \vec{x}, \vec{y} \rangle$ for every real number α.

Hints

1.5 (d) "The union of any collection of open sets" means $\cup_{\lambda \in I} A_\lambda$ where I is an index set (might be infinite). This is a quite standard fact in point-set topology. If you have trouble with it, I suggest you pick up a textbook on the subject and have a review. You have the simiar situation in 1.6 (d).

These two are equivalent via the fact:

$$\left(\bigcup_{\lambda \in I} A_\lambda\right)^c = \bigcap_{\lambda \in I} A_\lambda^c$$

where A_λ^c is the complement of A_λ.

1.10 (a) Prove

$$A + B = \bigcup_{b \in B} (A + b)$$

and $A + b$ is open and then apply 1.5 (d).

1.10 (b) Consider two sets $A = \{1, 2, 3, ..., n, ...\}$ and $B = \{1/2 - 1, 1/4 - 2, 1/8 - 3, ..., 2^{-n} - n, ...\}$ in $\mathbb{R}^1$. Show that both A and B are closed but $A + B$ is not.

2.2 (b) Look at (a) in Figure 2.7. Think of it as the union of two triangles.

A2. Let λ be a real number. Then

$$\langle \vec{x} + \lambda \vec{y}, \vec{x} + \lambda \vec{y} \rangle = ||\vec{x}||^2 + 2\lambda \langle \vec{x}, \vec{y} \rangle + \lambda^2 ||\vec{y}||^2$$

Observe the left hand side is nonnegative for every λ. Therefore, the discriminant of the right hand side (as a quadratic polynomial in λ) must be ≤ 0.