A6.1 Let $A$ and $B$ be two $n \times n$ real matrices satisfying $AB = BA$. Show that

$$f(A)g(B) = g(B)f(A)$$

for all polynomials $f(x)$ and $g(x) \in \mathbb{R}[x]$.

Proof. Since $AB = BA$, $A^iB^j = B^jA^i$ for all non-negative integers $i$ and $j$. Let

$$f(x) = \sum_{i=0}^{m} a_i x^i \text{ and } g(x) = \sum_{j=0}^{n} b_j x^j.$$ 

Then

$$f(A)g(B) = \left( \sum_{i=0}^{m} a_i A^i \right) \left( \sum_{j=0}^{n} b_j B^j \right) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_i b_j A^i B^j$$

$$= \sum_{i=0}^{m} \sum_{j=0}^{n} a_i b_j B^j A^i = \left( \sum_{j=0}^{n} b_j B^j \right) \left( \sum_{i=0}^{m} a_i A^i \right)$$

$$= g(B)f(A).$$

□

A6.2 Let $A$ be an $n \times n$ complex matrix with 0 as the only eigenvalue and let $f(x)$ be the minimal polynomial of $A$ with leading coefficient 1. Do the following:

(a) Show that $f(x) = x^m$ for some positive integer $m$.
(b) Show that $m \leq n$ in part (a).

Proof. Since 0 is the only eigenvalue of $A$, the characteristic polynomial of $A$ must be $\det(xI - A) = x^n$. By Cayley-Hamilton, $f(x)$ divides $x^n$. So $f(x) = x^m$ for some $m \leq n$. □

A6.3 Construct the following matrices with the given minimal polynomials. You must justify your answer.

(a) A $2 \times 2$ matrix with minimal polynomial $(x - 1)^2$.
(b) A $3 \times 3$ matrix with minimal polynomial $(x - 1)^2$.
(c) A $4 \times 4$ matrix with minimal polynomial $x(x - 1)(x - 2)$.
(d) A $4 \times 4$ matrix with minimal polynomial $x^2(x - 1)(x - 2)$.

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1. [http://www.math.ualberta.ca/~xichen/math32517w/hw6sol.pdf](http://www.math.ualberta.ca/~xichen/math32517w/hw6sol.pdf)
Solution. We construct Jordan matrices with the designated minimal polynomials.

(a) $A$ has only eigenvalues $1$. So

$$A = \begin{bmatrix} J_{1,a_1} & & \\ & J_{1,a_2} & \\ & & \ddots \\ & & & J_{1,a_k} \end{bmatrix}$$

with $a_1 + a_2 + \ldots + a_k = 2$. The minimal polynomial $A$ is

$$f(x) = \text{lcm}((x - 1)^{a_1}, (x - 1)^{a_2}, \ldots, (x - 1)^{a_k}) = (x - 1)^a$$

where $a = \max(a_1, a_2, \ldots, a_k)$. So we need to find positive integers $a_1, a_2, \ldots, a_k$ such that

$$a_1 + a_2 + \ldots + a_k = 2 \text{ and } \max(a_1, a_2, \ldots, a_k) = 2.$$ 

The only choice is $a_1 = 2$ and $k = 1$. So

$$A = J_{1,2} = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$$

has minimal polynomial $(x - 1)^2$.

(b) Similar to (a), we want to construct a Jordan matrix

$$A = \begin{bmatrix} J_{1,a_1} & & \\ & J_{1,a_2} & \\ & & \ddots \\ & & & J_{1,a_k} \end{bmatrix}$$

with

$$a_1 + a_2 + \ldots + a_k = 3 \text{ and } \max(a_1, a_2, \ldots, a_k) = 2.$$ 

We may choose $a_1 = 2$, $a_2 = 1$ and $k = 2$. So

$$A = \begin{bmatrix} J_{1,2} & \\ & J_{1,1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

has minimal polynomial $(x - 1)^2$.

(c) Since the minimal polynomial has no multiple roots, it is diagonalizable with eigenvalues $0, 1, 2$. So

$$A = \begin{bmatrix} 0 & \\ 0 & 1 \\ & 2 \end{bmatrix}$$

has minimal polynomial $x(x - 1)(x - 2)$. 
(d) Suppose that

\[ A = \begin{bmatrix}
J_{0,a_1} & & \\
& \ddots & \\
& & J_{0,a_k}
\end{bmatrix}
\]

Since the minimal polynomial of \( A \) is

\[
f(x) = \text{lcm}(x^{a_1}, \ldots, x^{a_k}, (x - 1)^{b_1}, \ldots, (x - 1)^{b_l}, \ldots, (x - 2)^{c_1}, \ldots, (x - 2)^{c_m})
\]

\[
x^a(x - 1)^b(x - 2)^c
\]

for \( a = \max(a_1, \ldots, a_k), b = \max(b_1, \ldots, b_l) \) and \( c = \max(c_1, \ldots, c_m) \), we need to find \( a_1, \ldots, a_k, b_1, \ldots, b_l, c_1, \ldots, c_m \) satisfying

\[
a_1 + \ldots + a_k + b_1 + \ldots + b_l + c_1 + \ldots + c_m = 4
\]

\[
\max(a_1, \ldots, a_k) = 2
\]

\[
\max(b_1, \ldots, b_l) = 1
\]

\[
\max(c_1, \ldots, c_m) = 1.
\]

So we may choose \( a_1 = 2 \) and \( b_1 = c_1 = k = l = m = 1 \) and

\[
A = \begin{bmatrix}
J_{0,2} & \\
& J_{1,1}
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
0 & 1
\end{bmatrix}
\]

has minimal polynomial \( x^2(x - 1)(x - 2) \).

\[ \square \]

A6.4 Let \( P \in M_{n \times n}(\mathbb{R}) \) and let \( T : M_{n \times n}(\mathbb{R}) \to M_{n \times n}(\mathbb{R}) \) be the linear endomorphism given by

\[ T(A) = AP \text{ for } A \in M_{n \times n}(\mathbb{R}). \]

Show that a polynomial \( f(x) \) is a minimal polynomial of \( T \) if and only if it is a minimal polynomial of \( P \).
Proof. For every polynomial \( f(x) = a_0 + a_1 x + \ldots + a_d x^d \) and \( A \in M_{n \times n}(\mathbb{R}) \),

\[
f(T)(A) = \sum_{i=0}^{d} a_i T^i(A) = \sum_{i=0}^{d} a_i AP^i
= A \sum_{i=0}^{d} a_i P^i = Af(P).
\]

If \( f(P) = 0 \), then \( f(T)(A) = Af(P) = 0 \) for all \( A \) and hence \( f(T) = 0 \). If \( f(T) = 0 \), then \( Af(P) = f(T)(A) = 0 \) for all \( A \); in particular, if \( A = I \), \( f(P) = Af(P) = 0 \). Therefore, \( f(T) = 0 \) if and only if \( f(P) = 0 \) for \( f(x) \in \mathbb{R}[x] \). So \( f(x) \) is a minimal polynomial of \( T \) if and only if it is a minimal polynomial of \( P \). \( \square \)

A6.5 Let \( V \) be a vector space over \( \mathbb{C} \) with a basis \( \{v_1, v_2, \ldots, v_n\} \) and let \( T : V \rightarrow V \) be a linear endomorphism given by

\[
T(v_1) = v_2, \ T(v_2) = v_3, \ldots, T(v_{n-1}) = v_n, \ T(v_n) = v_1.
\]

Do the following:

(a) Find a minimal polynomial of \( T \). You must justify your answer.

(b) Show that \( T \) is diagonalizable.

Proof. We may write \( T(v_k) = v_{k+1} \) for \( i = 1, 2, \ldots, n \), where we set \( v_a = v_b \) if \( a - b \) is divisible by \( n \). Then

\[
T^m(v_k) = T^{m-1}(T(v_k)) = T^{m-1}(v_{k+1})
= \ldots = v_{k+m}
\]

for all integers \( k \) and \( m \). Therefore,

\[
T^n(v_k) = v_{k+n} = v_k
\]

for all \( k \). For every \( v \in V \), \( v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n \) for some \( c_1, c_2, \ldots, c_n \in \mathbb{C} \) and

\[
T^n(v) = T^n(c_1 v_1 + c_2 v_2 + \ldots + c_n v_n)
= c_1 T^n(v_1) + c_2 T^n(v_2) + \cdots + c_n T^n(v_n)
= c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = v.
\]

Therefore, \( T^n = I \), i.e., \( f(T) = T^n - I = 0 \) for \( f(x) = x^n - 1 \).

We claim that \( f(x) = x^n - 1 \) is a minimal polynomial of \( T \). If it is not, there exists a nonzero polynomial

\[
g(x) = a_0 + a_1 x + \cdots + a_m x^m
\]
of degree $\deg g(x) = m < n$ such that $g(T) = 0$. Then

$$0 = g(T)(v_1) = (a_0I + a_1T + \cdots + a_mT^m)(v_1)$$

$$= \sum_{k=0}^{m} a_k T^k(v_1) = \sum_{k=0}^{m} a_k v_{k+1}$$

$$= a_0v_1 + a_1v_2 + \cdots + a_m v_{m+1}.$$ 

Since $m < n$, $v_1, v_2, \ldots, v_{m+1}$ are linearly independent. So $a_0 = a_1 = \ldots = a_m = 0$ and $g(x) = 0$. Contradiction. So $x^n - 1$ is a minimal polynomial of $T$.

To show that $T$ is diagonalizable, it suffices to show that $f(x) = x^n - 1$ has no multiple roots over $\mathbb{C}$. It suffices to verify that $\gcd(f(x), f’(x)) = 1$. Using Euclidean algorithm, we have

$$\gcd(f(x), f’(x)) = \gcd(x^n - 1, nx^{n-1})$$

$$= \gcd(-1, nx^{n-1}) = 1.$$ 

So the minimal polynomial $x^n - 1$ of $T$ has no multiple roots and hence $T$ is diagonalizable. \qed